Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology and its Applications 153 (2005) 493–527
Topology and its Applications
www.elsevier.com/locate/topol

Non-additivity of strong homology

Andrei V. Prasolov

Institutt for Matematikk og Statistikk, Universitetet i Tromsø, N-9037 Tromsø, Norway

Received 31 January 2003; accepted 26 December 2003

Abstract

A counter-example is constructed that shows that *neither* higher inverse limits of pro-groups *nor* strong homology of topological spaces are *additive*. The previous counter-example by S. Mardešić and A. Prasolov depended on the Continuum Hypothesis. The approach developed in this paper is applied also to that example in order to calculate the cardinality of the corresponding strong homology groups. It appeared that the above cardinality depends essentially on a set-theoretic model: it is *hypercontinuum* under a weaker version of the Continuum Hypothesis, and *zero* under the Proper Forcing Axiom.

MSC: primary 55N07, 55P55, 55U99, 18E25; secondary 55N40, 55U99

Keywords: Strong homology; Strong shape; Pro-groups; Dominating number; Higher inverse limits; Continuum Hypothesis

0. Introduction

The purpose of this paper is to prove that the strong homology $\overline{H}_p(X, G)$ is *not additive*. This result was announced by the author quite long ago, but the proof existed only in the form of a preprint [19]. In the meantime the author has succeeded to simplify the proof. Moreover, the machinery developed in this paper helped to simplify the proof of the main result in [18].

E-mail address: andreip@math.uit.no (A.V. Prasolov).
URL: http://www.math.uit.no/users/andreip/.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2003.12.022

The counter-example to the additivity is more complicated than the main example in [14]. The "old" example was a finite-dimensional metric space while the "new" one is paracompact but not metrizable. However, that old counter-example depended on the Continuum Hypothesis, while the new one is valid in *any* model of ZFC (Zermelo–Fraenkel axioms plus the Axiom of Choice). The corresponding pro-group is also more complicated than the pro-group $\mathbf{P}^{\omega}(\mathbb{Z})$ (see Definition 3.2) from the old example.

In order to accomplish the purpose, one needs to prove that the higher inverse limit functors $\mathbf{lim}^1$ and $\mathbf{lim}^2$ are not additive in the category **pro**-**AB** of Abelian pro-groups.

Remark 0.1. Inverse limit functor **lim** is additive, see Proposition 2.3.

Remark 0.2. The Čech homology $\check{H}$ is additive, see Proposition 2.2.

It was proven in [14] under the Continuum Hypothesis assumption that strong homology $\overline{H}_p(X,\mathbb{Z})$ is not additive. One of the obstructions to the additivity appeared to be $\mathbf{lim}^1\mathbf{P}^{\omega}(\mathbb{Z})$ for a certain pro-group $\mathbf{P}^{\omega}(\mathbb{Z})$. The obstruction was proven to be non-trivial under the above assumption. Later in [3] it was shown that $\mathbf{lim}^1\mathbf{P}^{\omega}(\mathbb{Z})$ is not zero under a weaker "$d=\aleph_1$" assumption, and that $\mathbf{lim}^1\mathbf{P}^{\omega}(\mathbb{Z})$ could be zero in some other set-theoretic models. The author in [17] investigated the problem of additivity for $\overline{H}_p(X,G)$ in the case of an arbitrary group G. Under the "$d=\aleph_1$" assumption he estimated the cardinality of $\mathbf{lim}^1\mathbf{P}^{\omega}(G)$:

$$|G|^{\aleph_0} \leqslant \left|\mathbf{lim}^1\mathbf{P}^{\omega}(G)\right| \leqslant |G|^{\aleph_1}$$

[17, Theorem 3]. However, in other set-theoretic models

$$\mathbf{lim}^1\mathbf{P}^{\omega}(G)=\mathbf{0}$$

for any finite or countable group G [17, Theorem 4]. In [18] the above cardinality estimate was significantly improved:

$$\left|\mathbf{lim}^1\mathbf{P}^{\omega}(G)\right| = |G|^{\aleph_1}.$$

In this paper we are also giving a simplified proof of that equality.

0.1. Preliminaries

We consider only Abelian groups in this paper, and therefore "group" will mean "Abelian group" and "pro-group" will mean "Abelian pro-group", i.e., an object of the pro-category **pro**-**AB**. We will follow the modern tradition of using symbols

$$\mathbf{lim},\ \mathbf{colim},\ \mathbf{lim}^n,\ \mathbf{colim}_n$$

instead of

$$\varprojlim,\ \varinjlim,\ \varprojlim{}^n,\ \varinjlim{}_n$$

for inverse (direct) limits and their right (left) derived functors.

Strong homology groups $\overline{H}_p(X,A;G)$, $p\geqslant 0$, for arbitrary pairs of topological spaces (X,A), were constructed by Yu.T. Lisica and S. Mardešić (see [11, Ch. 19], and the original papers [6,7,5,8]). These groups have many desired properties. They satisfy, in particular, all Eilenberg–Steenrod axioms for pairs (X, A) where X is paracompact, and A is closed in X (more generally, for all normal pairs, see [11, Ch. 19.1–19.2]). The groups $\overline{H}_p(X, A; G)$ are strong shape invariants ([5] and [11, Ch. 19.1]), and are trivial provided p is greater than the shape dimension $\mathbf{sd}\, X$ [11, Ch. 19.4]. Moreover, those strong homologies satisfy the strong excision axiom ([2] and [11, Ch. 19.2]) for normal pairs, and the cluster axiom for arbitrary spaces (see [22] for paracompact and [2] or [11, Ch. 19.3] for arbitrary spaces). Following J. Milnor [12], we shall call a homology theory h_* *additive* iff for any family of topological spaces $(X^\alpha\colon \alpha \in A)$ the canonical embeddings

$$j^\alpha : X^\alpha \to \coprod_{\alpha \in A} X^\alpha$$

induce isomorphisms

$$\bigoplus_{\alpha \in A} h_p(X^\alpha) \to h_p\left(\coprod_{\alpha \in A} X^\alpha\right)$$

(here $\coprod$ means the topological coproduct of spaces).

We use in the paper Miminoshvili's version ([13] and [11, Ch. 17]) of strong homologies that allows groups $\overline{H}_p$ with negative p, and coincides with the homologies from [6,7,5] for $p > 0$. Let us define an obstruction to the additivity homomorphism for strong homology by:

$$S_{p,X,A,G} := \mathbf{coker}\left(\varphi_{p,X,A,G} : \bigoplus_{\alpha \in A} \overline{H}_p(X^\alpha, G) \to \overline{H}_p\left(\coprod_{\alpha \in A} X^\alpha, G\right)\right)$$

where $(X^\alpha\colon \alpha \in A)$ is the family consisting of A copies of a space X. The obstruction to the corresponding homomorphism for pro-groups will be denoted by:

$$L_{p,\mathbf{C},A} := \mathbf{coker}\left(\psi_{p,\mathbf{C},A} : \bigoplus_{\alpha \in A} \mathbf{lim}^p \mathbf{C}^\alpha \to \mathbf{lim}^p\left(\bigoplus_{\alpha \in A} \mathbf{C}^\alpha\right)\right)$$

where $(\mathbf{C}^\alpha; \alpha \in A)$ is the family consisting of A copies of a pro-group $\mathbf{C}$.

Remark 0.3. Both φ and ψ are always injective, see Theorem 2.1.

Remark 0.4. $\psi_{0,\mathbf{C},A}$ are bijective for any $\mathbf{C}$ and A, see Proposition 2.3.

Let $Y(k)$ be the old counter-example (see Section 3). It was proven in [14] that

$$S_{k-1,Y(k),\omega,\mathbb{Z}} \neq 0$$

if the Continuum Hypothesis is assumed. It happens, however, that it is provable, assuming the Proper Forcing Axiom (PFA), that the group $S_{k-1,Y(k),\omega,\mathbb{Z}}$ is trivial [3]. It is proven in this paper (Theorem 2.5) that for any non-trivial countable Abelian group G the statement

$$S_{k-1,Y(k),\omega,G} = 0$$

is independent of the ZFC.

0.2. Agreements

From now on "countable set" will mean "infinite countable set".

The symbols

$$\omega = \omega_0, \aleph_0, \omega_1, \aleph_1, \omega_2, \aleph_2$$

and so on will mean "the first infinite ordinal (cardinal)", "the first uncountable ordinal (cardinal)", "the second uncountable ordinal (cardinal)" respectively. Thus the statement $c = \aleph_1$ is nothing else but the Continuum Hypothesis. It is popular to construct ordinals $\alpha, \beta, \ldots$ in such a way that the following conditions are equivalent:

$$\alpha < \beta \iff \alpha \in \beta \iff \alpha \subset \beta.$$

The last condition $\alpha \subset \beta$ will always mean

$$(\alpha \subseteq \beta)\ \&\ (\alpha \neq \beta).$$

If the ordinals are constructed that way, then an ordinal α coincides with the interval

$$[0, \alpha\rangle = \{\beta\colon\ 0 \leqslant \beta < \alpha\}.$$

We denote as usual the cardinality of the set S by $\mathbf{card}(S)$, or simply by $|S|$.

Call a monotone map $f : \Lambda \to \Lambda'$ between two partially ordered sets *cofinal* if $f(\Lambda)$ is cofinal in Λ', that is

$$(\forall \lambda' \in \Lambda')(\exists \lambda \in \Lambda)\big(f(\lambda) \geqslant \lambda'\big).$$

By *cofinality* of the ordinal α we shall mean the least ordinal β possessing a cofinal mapping $f : [0, \beta\rangle \to [0, \alpha\rangle$.

In [21] two cardinals b (*bounding* number), and d (*dominating* number) have been defined. Let $[0, \omega\rangle^{[0,\omega\rangle}$ be the set of all maps $[0, \omega\rangle \to [0, \omega\rangle$ with the componentwise ordering, and let $\ll$ be a new ordering on $[0, \omega\rangle^{[0,\omega\rangle}$ defined as follows:

$$\alpha \ll \beta \iff (\exists n_0 < \omega)(\forall n \geqslant n_0)\big(\alpha(n) \leqslant \beta(n)\big).$$

We denote by Λ' the pre-ordered set $([0, \omega\rangle^{[0,\omega\rangle}, \ll)$. Let λ_1 be its unbounded subset of minimal cardinality and λ_2 be its cofinal subset of minimal cardinality. The cardinals b and d are defined as follows:

$$b = \mathbf{card}(\lambda_1) \quad \text{and} \quad d = \mathbf{card}(\lambda_2).$$

It is known [21, §3] that

$$\aleph_1 \leqslant b \leqslant d \leqslant c.$$

Moreover, for any integers $1 \leqslant p \leqslant l \leqslant m$ there exists a model of ZFC, in which the following is true:

$$(b = \aleph_p)\&(d = \aleph_l)\&(c = \aleph_m).$$

It means that the statement above is consistent with the axioms ZFC (see [21, §5]).

For a category $\mathbf{C}$, let $|\mathbf{C}|$ denote its class of objects. Given

$$X, Y \in |\mathbf{C}|,$$

let $\mathbf{C}(X, Y)$ denote the set of morphisms $X \to Y$ in $\mathbf{C}$.

Definition 0.5. Let Λ be a pre-ordered set, and $\mathbf{C}$ be a category. A Λ-indexed inverse system will be a triple

$$X = (X_\lambda, p_{\lambda\mu}, \Lambda), \quad \lambda, \mu \in \Lambda,$$

where X_λ are objects of $\mathbf{C}$ and

$$p_{\lambda\mu} : X_\mu \to X_\lambda, \quad \lambda \leqslant \mu,$$

is a family of morphisms in $\mathbf{C}$ with

$$p_{\lambda\mu} \circ p_{\mu v} = p_{\lambda\mu}, \qquad p_{\lambda\lambda} = \mathrm{Id}_{X_\lambda}$$

for any $\lambda \leqslant \mu \leqslant \nu$. In the case $\mathbf{C} = \mathbf{AB}$ (the category of Abelian groups) or $\mathbf{C} = \mathbf{TOP}$ (the category of topological spaces) we shall speak of inverse systems of Abelian groups and topological spaces respectively. Let $\mathbf{C}^\Lambda$ (for example $\mathbf{AB}^\Lambda$ or $\mathbf{TOP}^\Lambda$) denote the category of Λ-indexed inverse systems in $\mathbf{C}$.

Given a category $\mathbf{C}$, let $\mathbf{Pro}$-$\mathbf{C}$ denote the corresponding pro-category [1]. Objects of this new category are inverse systems

$$\mathbf{A} = (A_\lambda, p_{\lambda\mu}, \Lambda)$$

where Λ is a directed partially ordered set. Given two such systems

$$\mathbf{A} = (A_\lambda, p_{\lambda\mu}, \Lambda) \quad \text{and} \quad \mathbf{B} = (B_\lambda, q_{\lambda\mu}, \Lambda'),$$

define

$$\mathbf{Pro}\text{-}\mathbf{C}(\mathbf{A}, \mathbf{B}) := \underset{\lambda' \in \Lambda'}{\mathbf{lim}}\ \underset{\lambda \in \Lambda}{\mathbf{colim}}\, \mathbf{C}(A_\lambda, B_{\lambda'}).$$

If $\mathbf{A} = (A_\lambda, p_{\lambda\mu}, \Lambda)$ is an inverse system in the category $\mathbf{C}$, then we shall denote by the same symbol $\mathbf{A}$ the corresponding pro-object in $\mathbf{Pro}$-$\mathbf{C}$, and by $\mathbf{lim}^s \mathbf{A}$ the sth right derived functor of the inverse limit functor $\mathbf{lim}$ [4]. Thus $\mathbf{lim}^0 = \mathbf{lim}$, and we set for convenience $\mathbf{lim}^s = 0$ for $s < 0$. It is well-known that the groups $\mathbf{lim}^s \mathbf{A}$ do not depend on the above representation of $\mathbf{A}$ by an inverse system. It follows, e.g., from [20, Theorem 2.1.10] (see also [23]).

A sequence

$$\mathbf{A} = (A_0 \xleftarrow{p} A_1 \xleftarrow{p} A_2 \xleftarrow{p} \cdots)$$

of objects and morphisms in the category $\mathbf{C}$ will be called a *tower*. If $\mathbf{A}$ is a tower of Abelian groups, then $\mathbf{lim}^s A = 0$ for $s > 1$, and $\mathbf{lim}^1 A = 0$ provided all p's are surjective [4, §2].

1. Main results

In this section we state the four main results (actually, two *pairs* of results) of the paper.

1.1. The new counter-example

Theorem 1. *For any non-trivial group G there exist pro-groups $\mathbf{A}(G)$ and $\mathbf{C}(G)$ such that the natural maps*

$$\bigoplus_{i<\omega} \mathbf{lim}^2 \mathbf{A}(G) \to \mathbf{lim}^2\Big(\bigoplus_{i<\omega} \mathbf{A}(G)\Big),$$

$$\bigoplus_{i<\omega} \mathbf{lim}^1 \mathbf{C}(G) \to \mathbf{lim}^1\Big(\bigoplus_{i<\omega} \mathbf{C}(G)\Big)$$

are injective, but not surjective.

See Section 6.1 for the proof.

Theorem 2. *There exists a paracompact space X such that for any non-trivial group G and any $p \geqslant -2$ the natural map*

$$\bigoplus_{i<\omega} \overline{H}_p(X, G) \to \overline{H}_p\Big(\coprod_{i<\omega} X, G\Big)$$

is injective but not surjective.

See Section 7.1 for the proof.

1.2. The old counter-example

Let $\mathbf{P}(G)$ be the pro-group from Definition 3.2.

Theorem 3. *Let*

$$c_p(G) := L_{p,\mathbf{P}(G),\omega} := \mathbf{coker}\Big(\psi_{p,\mathbf{P}(G),\omega} : \bigoplus_{\alpha<\omega} \mathbf{lim}^p \mathbf{P}(G) \to \mathbf{lim}^p\big(\mathbf{P}^\omega(G)\big)\Big).$$

Assume $d = \aleph_1$. Then

$$\big|c_1(G)\big| = |G|^{\aleph_1}.$$

See Section 5 for the proof.

According to Theorem 2.4(a) and Proposition 4.10 (see also Definition 3.1),

$$\begin{aligned} c_n(G) &= c_n(0, G) = c_n(1, G) = \cdots = c_n(k, G) = S_{k-n,Y(k),\omega,G} \\ &= \mathbf{coker}\Big(\varphi_{k-n,Y(k),\omega,G} : \bigoplus_{\alpha<\omega} \overline{H}_{k-n}\big(Y(k), G\big) \to \overline{H}_{k-n}\Big(\coprod_{\alpha<\omega} Y(k), G\Big)\Big). \end{aligned}$$

It was shown in [14], assuming the Continuum Hypothesis, that the statement

$$S_{k-1,Y(k),\omega,\mathbb{Z}} \neq 0$$

is valid for all $k \geqslant 0$. In [3] the same statement was proven under a weaker assumption $d = \aleph_1$. In the theorem below it is proven that under the last assumption the cardinality of $S_{k-1,Y(k),\omega,G}$ is large enough:

Theorem 4. *Assume $d = \aleph_1$ and $k \geqslant 0$. Then*

$$|S_{k-1,Y(k),\omega,G}| = |G|^{\aleph_1}.$$

Therefore the cardinality of the group $S_{k-1,Y(k),\omega,G}$, one of the obstructions to the additivity, is at least hypercontinuum for $G \neq 0$.

See Section 5 for the proof.

2. Other results

Theorem 2.1. *Both $\varphi_{p,X,A,G}$ and $\psi_{p,\mathbf{C},A}$ are monomorphisms for all p, X, A, G, and $\mathbf{C}$.*

See Section 4.1 for the proof.

Proposition 2.2. *The homomorphisms*

$$\bigoplus_{\alpha\in A} \check{H}_p(X^\alpha, G) \to \check{H}_p\Big(\coprod_{\alpha\in A} X^\alpha, G\Big)$$

are isomorphisms for all p and all families of topological spaces.

Proof. See [14, Theorem 9]. □

Proposition 2.3. *$\psi_{0,\mathbf{C},A}$ are isomorphisms for all X, A, and $\mathbf{C}$.*

Proof. Analogous to the proof of the above theorem. Actually, in the proof of [14, Theorem 9], the statement of Proposition 2.3 and Theorem 4.5 below were implicitly proven. □

Theorem 2.4. *Let*

$$\begin{aligned} c_n(k, G) &:= S_{k-n,Y(k),\omega,G} \\ &= \mathbf{coker}\Big(\varphi_{k-n,Y(k),\omega,G} : \bigoplus_{\alpha<\omega} \overline{H}_{k-n}\big(Y(k), G\big) \to \overline{H}_{k-n}\Big(\coprod_{\alpha<\omega} Y(k), G\Big)\Big). \end{aligned}$$

Then:

(a) *The groups $c_n(k, G)$ do not depend on k;*
(b) *$c_n(k, G) = 0$ for $n \leqslant 0$;*
(c) *Each group $c_n(k, G)$ is either trivial or infinitely generated.*

See Section 4.2 for the proof.

It is proven in [3] that the statement $c_1(\mathbb{Z})=0$ is consistent with the axioms ZFC. Thus it follows from [14] and [3] that the statement $c_1(\mathbb{Z})=0$ *does not depend* on the axioms ZFC. The theorem below is a generalization of that fact:

Theorem 2.5. *Let G be a countable or finite non-trivial Abelian group. Then the statement*

$$c_1(G)=0$$

does not depend on the axioms ZFC.

See Section 4.3 for the proof.

3. Constructions

Fix an integer $k \geqslant 0$. Let $Y(k)$ denote the k-dimensional "Hawaiian ear-ring", that is the wedge (or compact bouquet, or cluster) of countably many copies of the k-sphere S^k. One can consider $Y(k)$ as a subspace of $\mathbf{R}^{k+1}$ of the form

$$Y(k)=\bigcup_{i<\omega} S_i^k$$

where S_i^k is the sphere with the radius $1/i$ and the center at the point

$$\left(\frac{1}{i},0,0,\ldots,0\right)\in R^{k+1}.$$

Let $Y(k)^\omega$ be the topological sum (coproduct) of countably many copies of the space $Y(k)$:

$$Y(k)^\omega=\coprod_{i<\omega} Y(k).$$

The space $Y(k)^\omega$ can also be represented as a subspace of $\mathbf{R}^{k+1}$ consisting of subspaces which are homeomorphic to the space $Y(k)$.

The sequence of embeddings of $Y(k)$ into the topological coproduct $Y(k)^\omega$ induce homomorphisms ($n\in\mathbb{Z}$):

$$\varphi_{k-n,Y(k),\omega,G}:\bigoplus_{i<\omega}\overline{H}_{k-n}\big(Y(k),G\big)\to\overline{H}_{k-n}\big(Y(k)^\omega,G\big).$$

The kernels of these homomorphisms are trivial (see Theorem 2.1), while the cokernels serve as "obstructions" to the additivity axiom.

Definition 3.1. Let $c_n(k,G):=\mathbf{coker}\,\varphi_{k-n,Y(k),\omega,G}$.

Definition 3.2. Let $\mathbf{P}(G)$ be the tower of groups

$$\mathbf{P}(G):=(0\longleftarrow G\longleftarrow G\times G\longleftarrow G\times G\times G\longleftarrow\cdots);$$

and $\mathbf{P}^{\omega}(G)$ be the direct sum (coproduct) of pro-groups $\mathbf{P}(G)$ in the category **pro-AB**:

$$\mathbf{P}^{\omega}(G) := \bigoplus_{i<\omega} \mathbf{P}(G).$$

4. Higher limits and strong homology

A topological space X can be considered as a constant inverse system ($X_\lambda = X$, $p_{\lambda\mu} = \mathrm{Id}_X$). Let $\mathbf{X} = (X_\lambda, p_{\lambda\mu}, \Lambda)$ be an inverse system of spaces.

Definition 4.1. A morphism $q : X \to \mathbf{X}$ is called a strong expansion ([9] and [11, Ch. 7.1]) if q is a strong shape equivalence between pro-spaces (see [20, Definition 2.3.5]).

Definition 4.2. A strong expansion $q : X \to \mathbf{X}$ is called a strong ANR-expansion if all X_λ are ANRs.

Let $\mathbf{X} = (X_\lambda, p_{\lambda\mu}, \Lambda)$ be an inverse system of topological spaces. Then

$$C.(\mathbf{X}, G) = \big(C.(X_\lambda, G), (p_{\lambda\mu})_*, \Lambda\big)$$

will be an inverse system of chain complexes. Here $C.(X_\lambda, G) = C.(X_\lambda) \otimes G$, and $C.(X_\lambda)$ is the singular chain complex for X_λ. Let

$$\overline{H}_n(\mathbf{X}, G) := \overset{\infty}{H}_n\big(C.(\mathbf{X}, G)\big) := H_n\big(Tot\big(C.(\mathbf{X}, G)\big)\big)$$

be the strong homology of the inverse system $C.(\mathbf{X}, G)$ in the sense of Z. Miminoshvili [13]. Following S. Mardešić and Yu.T. Lisica, define strong homology of a space X as

$$\overline{H}_n(X, G) = \overline{H}_n(\mathbf{X}, G)$$

where $q : X \to \mathbf{X}$ is some strong ANR-expansion of X. It has been shown in [6,7,5] and [9] (see also [11, Ch. 19]) that this definition does not depend on the choice of the expansion q.

Remark 4.3. As was shown by Sibe Mardešić, it does not matter if strong homotopy and strong homology are defined using ANR-expansions, polyhedral expansions, or even $\mathcal{P}$-expansions where $\mathcal{P}$ is the class of spaces having homotopy type of an ANR. In this paper we will freely use all three types of these expansions.

The following theorem is proven in [16]:

Theorem 4.4. (a) *There exists a spectral sequence with the E_2 term*

$$E_2^{st} = \mathbf{lim}_\lambda^s H_{-t}(X_\lambda, G);$$

(b) *This sequence converges strongly to $\overline{H}_{-s-t}(X, G) = \overline{H}_{-s-t}(\mathbf{X}, G)$ provided $\lim^{\mathbf{1}}_{\mathbf{r}} E_r^{st} = 0$ for each s, t.*

Any strong polyhedral expansion $\mathbf{X}$ of a space X gives rise to the following pro-objects in the pro-category **pro-AB**:

$$\mathbf{pro}\text{-}H_n(X,G) := H_n(\mathbf{X},G) := \big(H_n(X_\lambda,G), H_n(p_{\lambda\mu}), \Lambda\big), \quad n < \omega,$$

which do not depend on the choice of the expansion $\mathbf{X}$. The above groups $H_n(X_\lambda,G)$ are singular homology groups of the polyhedra X_λ. Let X be the topological coproduct of the spaces X^α, $\alpha \in A$. Inclusions $i^\alpha : X^\alpha \to X$ induce homomorphisms

$$i_n : \bigoplus_{\alpha\in A} \mathbf{pro}\text{-}H_n(X^\alpha,G) \to \mathbf{pro}\text{-}H_n(X,G)$$

where the symbol $\oplus$ on the left side of the above formula means coproduct in the category **pro-AB** of Abelian pro-groups.

Theorem 4.5. *The homomorphisms i_n are isomorphisms in the category* **pro-AB**.

Proof. Choose for each $\alpha \in A$ a strong polyhedral expansion

$$q^\alpha : X^\alpha \to \mathbf{X}^\alpha = (X^\alpha_\lambda, p^\alpha_{\lambda\mu}, \Lambda^\alpha).$$

Let $\Lambda = \prod_{\alpha\in A} \Lambda^\alpha$ be a product in the category of partially ordered sets, that is

$$\Lambda = \left\{\lambda : A \to \bigcup_{\alpha\in A} \Lambda^\alpha :\ (\forall\alpha \in A)\big(\lambda(\alpha) \in \Lambda^\alpha\big)\right\}$$

with the following partial ordering:

$$\lambda \leqslant \mu \Longleftrightarrow (\forall\alpha \in A)\big(\lambda(\alpha) \leqslant \mu(\alpha)\big).$$

For $\lambda \in \Lambda$ denote

$$X_\lambda = \coprod_{\alpha\in A} X^\alpha_{\lambda(\alpha)}$$

and for $\lambda \leqslant \mu$ define a mapping $p_{\lambda\mu} : X_\mu \to X_\lambda$ which for every α coincides with the mapping $p^\alpha_{\lambda(\alpha),\mu(\alpha)}$, being restricted to $X^\alpha_{\mu(\alpha)}$. Let us finally define a morphism

$$q : X \to \mathbf{X} = (X_\lambda, p_{\lambda\mu}, \Lambda)$$

of inverse systems in such a way that for each α, the maps $q_\lambda : X \to X_\lambda$ restricted to X^α, coincide with the mappings

$$q^\alpha_{\lambda(\alpha)} : X^\alpha \to X^\alpha_{\lambda(\alpha)}. \qquad \square$$

Lemma 4.6. $q : X \to \mathbf{X}$ *is a strong polyhedral expansion.*

Proof. The above pro-space $\mathbf{X}$ is in fact the coproduct of the pro-spaces $\mathbf{X}^\alpha$ (compare with [20, proof of Proposition 2.4.1]). Apply [20, Corollary 2.4.6]. $\square$

Let us construct effectively the direct sum (coproduct) in the category of Abelian pro-groups **pro-AB**. See also [20, proof of Proposition 2.4.1]. Let $(\mathbf{G}^\alpha:\ \alpha \in A)$ be a family of Abelian pro-groups such that each $\mathbf{G}^\alpha$ is represented by an inverse system

$$\mathbf{G}^\alpha = (G^\alpha_\lambda, p^\alpha_{\lambda\mu}, \Lambda^\alpha).$$

Define the inverse system

$$\mathbf{G} = \bigoplus_{\alpha \in A} \mathbf{G}^{\alpha} = (G_\lambda, p_{\lambda\mu}, \Lambda)$$

where

$$\Lambda = \prod_{\alpha \in A} \Lambda^{\alpha}; \qquad G_\lambda = \bigoplus_{\alpha \in A} G^{\alpha}_{\lambda(\alpha)}, \quad \lambda \in \Lambda,$$

and the homomorphisms $p_{\lambda\mu}$ are induced by the homomorphisms $p_{\lambda(\alpha),\mu(\alpha)}$, $\alpha \in A$.

Lemma 4.7. *The pro-group* $\mathbf{G}$ *is a coproduct of the pro-groups* $\mathbf{G}^{\alpha}$ *in the category* **pro-AB**.

Proof. It follows from [20, proof of Proposition 2.4.1]. Let us give an alternative proof.

The projections

$$\pi^{\alpha} : \Lambda \to \Lambda^{\alpha} \quad \bigl(\pi^{\alpha}(\lambda) = \lambda(\alpha)\bigr)$$

are cofinal for each $\alpha \in A$, and therefore, the pro-group $\mathbf{H}^{\alpha}$, where

$$\mathbf{H}^{\alpha} = (G^{\alpha}_{\pi^{\alpha}(\lambda)}, p^{\alpha}_{\pi^{\alpha}(\lambda)\pi^{\alpha}(\mu)}, \Lambda),$$

is isomorphic to $\mathbf{G}^{\alpha}$. Moreover, each group $G^{\alpha}_{\pi^{\alpha}(\lambda)} = G^{\alpha}_{\lambda(\alpha)}$ is embedded canonically into the direct sum $G_\lambda = \bigoplus_{\alpha \in A} G^{\alpha}_{\lambda(\alpha)}$, which in turn defines a morphism $\mathbf{H}^{\alpha} \to \mathbf{G}$ of pro-groups and a corresponding morphism

$$i^{\alpha}\colon\ \mathbf{G}^{\alpha} \approx \mathbf{H}^{\alpha} \to \mathbf{G}$$

in the category **pro-AB**. It is easy to check that for any pro-group $\mathbf{B}$ the natural homomorphism

$$\xi : \mathbf{Hom}_{\mathbf{pro\text{-}AB}}(\mathbf{G}, \mathbf{B}) \to \prod_{\alpha \in A} \mathbf{Hom}_{\mathbf{pro\text{-}AB}}(\mathbf{G}^{\alpha}, \mathbf{B})$$

is bijective, and thus

$$\mathbf{G} \approx \bigoplus_{\alpha \in A} \mathbf{G}^{\alpha}$$

in **pro**-**AB**. □

To finish the proof of the theorem, compare the two constructions of coproducts in the categories **pro**-*TOP* and **pro-AB**, respectively, and use additivity of the singular homology.

4.1. Proof of Theorem 2.1

First consider the case of a finite index set A.

Proposition 4.8. *Let* A *be finite. Then for an arbitrary family*

$$(X^{\alpha}\colon\ \alpha \in A)$$

of spaces and for an arbitrary family $(\mathbf{C}^\alpha\colon\ \alpha \in A)$ *of pro-groups the mappings*

$$\bigoplus_{\alpha\in A} \overline{H}_p(X^\alpha, G) \to \overline{H}_p\Bigl(\coprod_{\alpha\in A} X^\alpha, G\Bigr)$$

and

$$\bigoplus_{\alpha\in A} \mathbf{lim}^p \mathbf{C}^\alpha \to \mathbf{lim}^p\Bigl(\bigoplus_{\alpha\in A} \mathbf{C}^\alpha\Bigr)$$

are isomorphisms.

Proof. The first statement follows from the Eilenberg–Steenrod axioms for strong homology.

The category **pro-AB** is additive, and even Abelian, and therefore finite coproducts in **pro-AB** coincide with finite products. Higher inverse limits, being *right* derived functors, commute with products, thus implying the second statement. □

Theorem 2.1 now follows from a more general

Proposition 4.9. *Let*

$$i_\alpha : X^\alpha \to X = \coprod_{\beta\in A} X^\beta$$

be a canonical imbedding of the α*th summand into the topological sum. Then the mapping*

$$\varphi = \bigoplus_{\alpha\in A} (i_\alpha)_* : \bigoplus_{\alpha\in A} \overline{H}_n(X^\alpha, G) \to \overline{H}_n(X, G)$$

is a monomorphism for all $n \in \mathbf{Z}$ *where* $(i_\alpha)_* = \overline{H}_n(i_\alpha, G)$.

Proof. Let $f_\alpha : X \to X^\alpha \amalg \{x_0\}$ be a map which coincides with i_α^{-1} on the subset $i_\alpha(X^\alpha) \subseteq X$, and maps $X - i_\alpha(X^\alpha)$ onto the point x_0, and let π_α be the composition

$$\overline{H}(X, G) \to \overline{H}\bigl(X^\alpha \amalg \{x_0\}, G\bigr) \approx \overline{H}(X^\alpha, G) \oplus \overline{H}\bigl(\{x_0\}, G\bigr) \to \overline{H}_n(X^\alpha, G)$$

of mappings, the first one being $\overline{H}_n(f_\alpha, G)$ and the other one being the canonical projection of the direct sum onto the first summand. It easy to see that

$$\pi_\alpha \circ (i_\beta)_* = \begin{cases} \mathrm{Id} & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

It follows therefore that the composition

$$\bigoplus_{\alpha\in A} \overline{H}_n(X^\alpha, G) \xrightarrow{\varphi} \overline{H}_n(X, G) \xrightarrow{\psi} \prod_{\alpha\in A} \overline{H}_n(X^\alpha, G),$$

where $\psi = \prod_{\alpha\in A} \pi_\alpha$, coincides with the canonical imbedding of the direct sum into the direct product, and thus φ is a monomorphism. □

4.2. Proof of Theorem 2.4

Using the spectral sequence from Theorem 4.4, we shall compute strong homology of the spaces $Y(k)$ and $Y(k)^\omega$. We first choose a strong ANR-expansion $q(k): Y(k) \to \mathbf{Y}(k)$ for $Y(k)$. Let $\Lambda = [0, \omega\rangle$, and for $n < \omega$ let $Y(k)_n$ be the wedge of n copies of the k-sphere S^k:

$$Y(k)_n = \bigvee^n S^k.$$

For $m \leqslant n$ define mappings

$$p_{mn}: Y(k)_n \to Y(k)_m$$

which map the first m spheres of the first wedge identically onto the corresponding spheres of the second one, and maps the other $n - m$ spheres to the base point. Let

$$\mathbf{Y}(k) = \bigl(Y(k)_m, p_{mn}, \omega\bigr) \in \mathbf{pro}\text{-}\mathit{TOP}.$$

The mappings $q(k)_n : Y(k) \to Y(k)_n$, $n < \omega$, are defined analogously (all spheres but the first n, are mapped to the base point), and give rise to a mapping

$$q(k): Y(k) \to \mathbf{Y}(k)$$

from the trivial pro-space $Y(k)$ to the pro-space $\mathbf{Y}(k)$. It is easy to see that the mapping

$$\mathbf{lim}\bigl(q(k)\bigr): Y(k) \to \mathbf{lim}\bigl(\mathbf{Y}(k)\bigr)$$

is a homeomorphism, and therefore q is a strong expansion, because all $Y(k)_m$ are compact metric spaces. We calculate, using the strong expansion q, the pro-homology of $Y^{(k)}$:

$$\mathbf{pro}\text{-}H_n(Y^{(k)}, G) = \begin{cases} \mathbf{P}(G), & n = k, \\ G, & n = 0 \neq k, \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{P}(G)$ is the pro-group given by the following tower:

$$\mathbf{P}(G) = (0 \leftarrow G \leftarrow G \times G \leftarrow G \times G \times G \leftarrow \cdots).$$

Furthermore, $\mathbf{lim}^s \mathbf{P}(G) = 0$ for $s \geqslant 1$ because the tower $\mathbf{P}(G)$ consists of epimorphisms, and therefore the spectral sequence E_r^{st} degenerates, and we get:

$$\overline{H}_n\bigl(Y(k), G\bigr) = \check{H}_n\bigl(Y(k), G\bigr) = \begin{cases} \prod_{i<\omega} G, & n = k, \\ G, & n = 0 \neq k, \\ 0 & \text{otherwise} \end{cases}$$

where $\check{H}_*$ is the Čech homology. In fact, the above formula follows also from the cluster axiom for strong homology.

Let us calculate now the pro-groups $\mathbf{pro}\text{-}H_n(Y(k)^\omega, G)$ and the strong homology groups $\overline{H}_n(Y(k)^\omega, G)$.

It follows from Theorem 4.5 that

$$\mathbf{pro}\text{-}H_n\bigl(Y(k)^\omega, G\bigr) = \begin{cases} \mathbf{P}^\omega(G) = \bigoplus_{i<\omega} \mathbf{P}(G), & n = k, \\ \bigoplus_{i<\omega} G, & n = 0 \neq k, \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{P}^\omega(G) = (P^\omega_\lambda, p_{\lambda\mu}, \Lambda)$ is the following inverse system of Abelian groups. The partially ordered set Λ is the set $[0,\omega\rangle^{[0,\omega\rangle}$ of functions $[0,\omega\rangle \to [0,\omega\rangle$ with the componentwise ordering:

$$\lambda \leqslant \mu \Longleftrightarrow (\forall i < \omega)\bigl(\lambda(i) \leqslant \mu(i)\bigr).$$

We set further for $\lambda \in \Lambda$:

$$P^\omega_\lambda = \bigoplus_{i<\omega}\biggl(\bigoplus_{0\leqslant j\leqslant\lambda(i)} G\biggr) = \bigoplus_{U_\lambda} G$$

where

$$U_\lambda = \bigl\{(i,j) < \omega\times\omega\colon\ j \leqslant \lambda(i)\bigr\}.$$

Let $p_{\lambda\mu} : P^\omega_\mu \to P^\omega_\lambda$, $\lambda \leqslant \mu$, be the projections induced by the inclusions $U_\lambda \subseteq U_\mu$.

Proposition 4.10. *The strong homology of $Y(k)^\omega$ is given by the following formula*:

$$\overline{H}_n\bigl(Y(k)^\omega, G\bigr) = \begin{cases} \mathbf{lim}(\mathbf{P}^\omega(G)) \approx \bigoplus_{i<\omega}(\prod_{j<\omega} G), & n = k,\\ \mathbf{lim}^{k-n}\mathbf{P}^\omega(G), & 0 \neq n < k,\\ 0, & n > k,\\ \mathbf{lim}^k\mathbf{P}^\omega(G) \oplus (\bigoplus_{i<\omega} G), & n = 0 \neq k. \end{cases}$$

Proof. Consider the spectral sequence of Theorem 4.4 for the space $Y(k)^\omega$. In the case $k = 0$ the spectral sequence degenerates because

$$E_2^{st} = \begin{cases} \lim^s \mathbf{P}^\omega(G), & t = 0,\\ 0, & t \neq 0, \end{cases}$$

and it follows that

$$\overline{H}_n\bigl(Y(0)^\omega, G\bigr) \approx E_\infty^{-n,0} \approx E_2^{-n,0}.$$

In the case $k \neq 0$ the non-zero elements of the spectral sequence lie on the line $s = -k$ and at the point $(0,0)$. Theorem 4.4 gives us the desired isomorphisms

$$\overline{H}_n\bigl(Y(k)^\omega, G\bigr) \approx E_\infty^{k-n,-k} \approx E_2^{k-n,-k} \approx \mathbf{lim}^{k-n}\mathbf{P}^\omega(G)$$

for $n \neq 0, 1$, and an exact sequence

$$0 \to E_2^{k,-k} \to \overline{H}_0\bigl(Y(k)^\omega, G\bigr) \xrightarrow{\psi} E_2^{0,0} \to E_2^{k+1,-k} \to \overline{H}_{-1}\bigl(Y(k)^\omega, G\bigr) \to 0.$$

However, ψ is a splitting epimorphism with the left inverse $\varphi_{0,Y(k),\omega,G}$ from Section 1 ($\psi \circ \varphi = \mathrm{Id}$), and thus,

$$\overline{H}_0(X^{(k)}, G) \approx E_2^{k,-k} \oplus E_2^{0,0} \approx \lim^k \mathbf{P}^\omega(G) \oplus \biggl(\bigoplus_{i<\omega} G\biggr),$$

$$\overline{H}_{-1}(X^{(k)}, G) \approx E_2^{k+1,-k} \approx \lim^{k+1} \mathbf{P}^\omega(G)$$

as desired. □

Now we see that

$$c_n(k,G)=\mathbf{coker}\,\varphi_{k-n,Y(k),\omega,G}\approx\begin{cases}\mathbf{lim}^n\mathbf{P}^\omega(G), & n>0,\\ 0, & n\leqslant 0,\end{cases}$$

which proves (a) and (b) of Theorem 2.4. In order to prove (c), it is sufficient to construct morphisms

$$\varphi_i,\psi_i:\mathbf{P}^\omega(G)\to\mathbf{P}^\omega(G),\quad i<\omega,$$

such that

$$\psi_i\circ\varphi_j=\delta_{ij}\mathrm{Id}:\mathbf{P}^\omega(G)\to\mathbf{P}^\omega(G).$$

Then the composition

$$A=\bigoplus_{i<\omega}\mathbf{lim}^s\mathbf{P}^\omega(G)\xrightarrow{\bigoplus\mathbf{lim}^s(\varphi_i)}\mathbf{lim}^s\mathbf{P}^\omega(G)\xrightarrow{\prod\mathbf{lim}^s(\psi_j)}\prod_{i<\omega}\mathbf{lim}^s\mathbf{P}^\omega(G)$$

will coincide with the canonical inclusion of the direct sum into the direct product. Suppose that $\mathbf{lim}^s\mathbf{P}^\omega(G)$ is a non-trivial and finitely generated Abelian group. The group A, being an infinite direct sum of non-trivial groups, cannot be finitely generated. At the same time A can be included into the finitely generated group $\mathbf{lim}^s\mathbf{P}^\omega(G)$, and therefore is finitely generated as well. This contradiction reduces the proof of (c) to the existence of morphisms φ_i,ψ_j above.

Let $\xi:[0,\omega\rangle\times[0,\omega\rangle\to[0,\omega\rangle$ be any 1–1 correspondence. We fix a $t<\omega$. If $\lambda\in\Lambda$, that is $\lambda:[0,\omega\rangle\to[0,\omega\rangle$, then

$$B_\lambda=\mathbf{P}^\omega(G)_\lambda=\bigoplus_{(i,j)\in U_\lambda}G=\bigl\{f:U_\lambda\to G:\ \text{almost all } f(i,j)\text{ equal zero}\bigr\}$$

("almost all" means "all but finitely many"). Let us define $\varphi_t(\lambda),\psi_t(\lambda):B_\lambda\to B_\lambda$ by the following:

$$\bigl[\varphi_t(\lambda)(f)\bigr](i,j)=f\bigl(\xi(t,i),j\bigr),$$

$$\bigl[\psi_t(\lambda)(f)\bigr](i,j)=\begin{cases}f(k,j), & i=\xi(t,k),\\ 0, & \text{otherwise.}\end{cases}$$

It follows that

$$\bigl[\bigl(\psi_s(\lambda)\circ\varphi_t(\lambda)\bigr)(f)\bigr](i,j)=\bigl[\varphi_t(\lambda)(f)\bigr]\bigl(\xi(s,i),j\bigr)=\begin{cases}f(i,j), & s=t,\\ 0, & s\neq t.\end{cases}$$

Therefore $\psi_s(\lambda)\circ\varphi_t(\lambda)=\delta_{st}\mathrm{Id}_{B_\lambda}$, where δ_{st} is the Kronecker symbol, and the systems $(\varphi_s(\lambda),\psi_s(\lambda):\ \lambda\in\Lambda)$ determine the desired morphisms $\varphi_s,\psi_s:\mathbf{P}^\omega(G)\to\mathbf{P}^\omega(G)$.

4.3. Proof of Theorem 2.5

In the case $G=\mathbb{Z}$, a model of ZFC was constructed in [3], in which the statement

$\mathbf{S}(G)=$ "$G^{[0,\omega\rangle\times[0,\omega\rangle}\to\mathbf{lim}(\mathbf{R}(G))$ is onto" is valid (see Definition 5.5). The statement $\mathbf{S}(G)$ is equivalent (Proposition 5.1) to $c_1(G) = 0$. Let DSV denote the model above. Note, however, that the definitions of $\mathbf{lim}(\mathbf{R}(G))$ and the mapping $G^{[0,\omega\rangle\times[0,\omega\rangle} \to \mathbf{lim}(\mathbf{R}(G))$ do *not depend* on the *group structure* on G. Thus in the model DSV the statement $\mathbf{S}(G)$ is true for an arbitrary infinite countable group G. Let G be a non-trivial finite or infinite countable Abelian group. Let further "$d = \aleph_1$" denote a model of ZFC, in which $d = \aleph_1$ is true. It follows from Theorem 4 that the group $c_1(G)$ has at least the cardinality of hypercontinuum in that model. There exists, however, a pair (i, p) of homomorphisms $G \rightleftarrows G \times \mathbb{Z}$ such that $p \circ i = \mathrm{Id}_G$. It follows from the diagram

$$i_* : c_1(G) \rightleftarrows c_1(G \times \mathbb{Z}) : p_*$$

that $p_* \circ i_* = \mathrm{Id}_{c_1(G)}$, that is $c_1(G)$ is isomorphic to a direct summand of $c_1(G \times \mathbb{Z})$. The last group is trivial in the model DSV, because $G \times \mathbb{Z}$ is an infinite countable group. Finally $c_1(G) = 0$ in the model DSV and $c_1(G) \neq 0$ in the model "$d = \aleph_1$". Thus the statement $\mathbf{S}(G)$ does not depend on the ZFC axioms, as desired.

5. Proof of Theorems 3 and 4

Let us consider a more general situation. Given an ordinal α, let $\mathbf{P}_\alpha(G)$ be the following pro-group:

$$\mathbf{P}_\alpha(G) = \big(\mathbf{P}_\alpha(G)_\beta, p_{\beta\gamma}, [0, \alpha\rangle\big)$$

where

$$\mathbf{P}_\alpha(G)_\beta = \bigoplus_{\delta\in[0,\beta\rangle} G$$

and

$$p_{\beta\gamma} : \mathbf{P}_\alpha(G)_\gamma = \bigoplus_{\delta\in[0,\gamma\rangle} G \to \mathbf{P}_\alpha(G)_\beta = \bigoplus_{\delta\in[0,\beta\rangle} G$$

for $\beta < \gamma$ are natural projections induced by inclusions $[0, \beta\rangle \subseteq [0, \gamma\rangle$. Let further S be an arbitrary set. Define $\Lambda = [0, \alpha\rangle^S$ with the componentwise ordering, and let

$$\mathbf{P}_\alpha(G)^S = \bigoplus_{i\in S} \mathbf{P}_\alpha(G) = \big(\big(\mathbf{P}_\alpha(G)^S\big)_\lambda, p_{\lambda\mu}, \Lambda\big)$$

be the coproduct of S copies of $\mathbf{P}_\alpha(G)$. A more detailed description follows:

$$\big(\mathbf{P}_\alpha(G)^S\big)_\lambda = \bigoplus_{i\in S}\Bigg(\bigoplus_{\delta\in[0,\lambda(i)\rangle} G\Bigg) = \bigoplus_{(i,\delta)\in U_\lambda} G$$

where

$$U_\lambda = \big\{(i, \delta) \in S \times [0, \alpha\rangle : \delta < \lambda(i)\big\}.$$

The homomorphisms $p_{\lambda\mu}$ are natural projections induced by inclusions $U_\lambda \subseteq U_\mu$.

We define two new Λ-indexed pro-groups

$$\mathbf{Q}_\alpha^S(G) = (Q_\lambda, p'_{\lambda\mu}, \Lambda), \qquad \mathbf{R}_\alpha^S(G) = (R_\lambda, p''_{\lambda\mu}, \Lambda)$$

by the following:

$$Q_\lambda = \prod_{(i,\delta)\in U_\lambda} G, \qquad \mathbf{R}_\alpha^S(G) = \mathbf{Q}_\alpha^S(G)/\mathbf{P}_\alpha(G)^S,$$

and $p'_{\lambda\mu}$ are natural projections.

Proposition 5.1.

$$\mathbf{lim}^s \mathbf{P}_\alpha(G)^S \approx \mathbf{lim}^{s-1} \mathbf{R}_\alpha^S(G), \quad s \geqslant 2;$$

$$\begin{aligned}\mathbf{lim}^1 \mathbf{P}_\alpha(G)^S &\approx \mathbf{coker}\big(\mathbf{lim}\big(\mathbf{Q}_\alpha^S(G)\big) \to \mathbf{lim}\big(\mathbf{R}_\alpha^S(G)\big)\big)\\ &\approx \mathbf{coker}\big(G^{S\times[0,\alpha\rangle} \to \mathbf{lim}\big(\mathbf{R}_\alpha^S(G)\big)\big).\end{aligned}$$

Proof. Consider the long exact sequence [4]

$$\begin{aligned}&0 \to \mathbf{lim}\big(\mathbf{P}_\alpha(G)^S\big) \to \mathbf{lim}\big(\mathbf{Q}_\alpha^S(G)\big) \to \mathbf{lim}\big(\mathbf{R}_\alpha^S(G)\big)\\ &\quad \to \mathbf{lim}^1 \mathbf{P}_\alpha(G)^S \to \mathbf{lim}^1 \mathbf{Q}_\alpha^S(G) \to \cdots\\ &\quad \to \mathbf{lim}^{s-1} \mathbf{R}_\alpha^S(G) \to \mathbf{lim}^s \mathbf{P}_\alpha(G)^S \to \cdots\end{aligned}$$

which follows from the short exact sequence

$$0 \to \mathbf{P}_\alpha(G)^S \to \mathbf{Q}_\alpha^S(G) \to \mathbf{R}_\alpha^S(G) \to 0$$

of inverse systems. The formulae for $\mathbf{lim}^s \mathbf{P}_\alpha(G)^S$ follow immediately from that long exact sequence and Lemma 5.2 below. □

Lemma 5.2. $\mathbf{lim}^s \mathbf{Q}_\alpha^S(G) = 0$ *for* $s \geqslant 1$, *and* $\mathbf{lim}(\mathbf{Q}_\alpha^S(G)) \approx G^{S\times[0,\alpha\rangle}$.

Proof. The group $\mathbf{lim}^s \mathbf{Q}_\alpha^S(G)$ is isomorphic [4, Theorem 4.1] to the sth cohomology group of the following cochain complex $C^\cdot$:

$$C^n = \prod_{\lambda=(\lambda_0\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n)} Q_{\lambda_0} \approx \prod_\lambda \prod_{i\in S} \Big(\prod_{\delta<\lambda_0(i)} G\Big) \approx \prod_{(i,\delta)\in S\times[0,\alpha\rangle} E^n(i,\delta)$$

where

$$E^n(i,\delta) = \prod_{\substack{\lambda_0\leqslant\cdots\leqslant\lambda_n\\ \delta<\lambda_0(i)}} G.$$

The coboundary operator d on $C^\cdot$ induces a coboundary operator on $E^\cdot$ such that $E^\cdot(i,\delta)$ will be cochain complexes for all i, δ, and the cochain complex $C^\cdot$ is isomorphic to the product of the cochain complexes $E^\cdot(i,\delta)$. Clearly

$$\mathbf{lim}^s \mathbf{Q}_\alpha^S(G) \approx H^s(C^\cdot) \approx \prod_{\substack{i\in S\\ \delta<\alpha}} H^s\big(E^\cdot(i,\delta)\big).$$

In order to calculate the groups $H^s(E^{\cdot}(i,\delta))$, let us consider the subset $\Gamma(i,\delta) \subseteq [0,\alpha\rangle^S$ which consists of all $\lambda \in [0,\alpha\rangle^S$ satisfying $0 \leqslant \delta < \lambda(i)$. Let $\mathbf{E}_{i\delta}(G)$ be the constant $\Gamma(i,\delta)$-indexed inverse system $\mathbf{E}_{i\delta}(G) \equiv G$. It follows from [4, Theorem 4.1] that

$$\mathbf{lim}^s \mathbf{E}_{i\delta}(G) \approx H^s\bigl(E^{\cdot}(i,\delta)\bigr).$$

The pro-group $\mathbf{E}_{i\delta}(G)$, however, is isomorphic to the constant pro-group G, and thus $\mathbf{lim}^s \mathbf{E}_{i\delta}(G) = 0$ for $s \geqslant 1$, and $\mathbf{lim}(\mathbf{E}_{i\delta}(G)) \approx G$. Now the desired formulae for $\mathbf{lim}^s \mathbf{Q}^S_\alpha(G)$ can easily be obtained. $\square$

The description of the inverse system $\mathbf{R}^S_\alpha(G)$ and Proposition 5.1 allow us to give the following description of $\mathbf{lim}^1 \mathbf{P}_\alpha(G)^S$. We say that two functions

$$f : U \to G \quad \text{and} \quad g : V \to G$$

almost agree iff the set

$$\bigl\{u \in U \cap V\colon\ f(u) \neq g(u)\bigr\}$$

is *finite*. The elements of $\mathbf{lim}(\mathbf{R}^S_\alpha(G))$ can be represented as families of almost agreeing functions

$$f_\lambda : U_\lambda \to G, \quad \lambda \in [0,\alpha\rangle^S.$$

We shall say that $(f_\lambda) \equiv (g_\lambda)$ if for all $\lambda \in [0,\alpha\rangle^S$ the functions f_λ and g_λ almost agree. A family $(f_\lambda\colon\ \lambda \in \Lambda)$ will be called *trivial* if there exists a function $f : S \times [0,\alpha\rangle \to G$ such that for every $\lambda \in \Lambda$ the functions f and f_λ almost agree. The statement below follows from Proposition 5.1.

Corollary 5.3. $\mathbf{lim}^1 \mathbf{P}_\alpha(G)^S$ *is isomorphic to the quotient group of "equivalent classes of families of almost agreeing functions" modulo "the group of trivial families".*

Remark 5.4. In the above Corollary one can easily substitute "*equivalent classes of families* of almost agreeing functions" by "*families* of almost agreeing functions". Indeed, if $(f_\lambda) \equiv (g_\lambda)$, then the constant function 0 almost agrees with $(f_\lambda - g_\lambda)$, and therefore that latter family is trivial.

Let Λ' be a partially ordered set $([0,\omega\rangle^{[0,\omega\rangle}, \ll)$ where $\ll$ is the pre-ordering introduced in Section 0.2. Let further $\pi : \Lambda \to \Lambda'$ be a monotonic map which is identical on $[0,\omega\rangle^{[0,\omega\rangle}$.

Definition 5.5. Let

$$\mathbf{R}(G) := \mathbf{R}^\omega_\omega(G).$$

Since $\mathbf{R}(G)$ does not "distinguish" the elements $\lambda \in [0,\omega\rangle^{[0,\omega\rangle}$ which are equivalent relative to the pre-ordering $\ll$, the inverse system for $\mathbf{R}(G)$ can be represented as $\mathbf{R}(G) = \mathbf{S} \circ \pi$ where $\mathbf{S}$ is a Λ'-indexed inverse system. The mapping π is cofinal, and therefore

$$\mathbf{lim}^s \mathbf{R}(G) \approx \mathbf{lim}^s \mathbf{S}, \quad s \geqslant 0.$$

Let d be the dominating number. There exists a cofinal map $g:\delta \to \Lambda'$ where δ is an ordinal of cardinality d. Let further $\mathbf{T}$ denote the δ-indexed system $\mathbf{S} \circ g$. Similarly we have

$$\mathbf{lim}^s \mathbf{R}(G) \approx \mathbf{lim}^s \mathbf{S} \approx \mathbf{lim}^s \mathbf{T}, \quad s \geqslant 0.$$

5.1. "Modulo finite" arithmetic

Let us fix an infinite set Z. Introduce the following binary relations on the set of all subsets of Z:

$$U \sqsubseteq V \iff \mathbf{card}(U - V) < \infty;$$

$$U \sim V \iff (U \sqsubseteq V)\&(V \sqsubseteq U);$$

$$\begin{aligned} U \sqsubset V &\iff (U \sqsubseteq V)\&(U \nsim V) \\ &\iff \bigl(\mathbf{card}(U - V) < \infty\bigr)\&\bigl(\mathbf{card}(V - U) = \infty\bigr); \end{aligned}$$

$$U \perp V \iff \mathbf{card}(U \cap V) < \infty;$$

$$U \perp (V_\alpha\colon \alpha \in A) \iff (\forall \alpha \in A)(U \perp V_\alpha);$$

$$(U_\alpha\colon \alpha \in A) \perp (V_\beta\colon \beta \in B) \iff (\forall \alpha \in A)(\forall \beta \in B)(U_\alpha \perp V_\beta);$$

$$U \rhd (V_\alpha\colon \alpha \in A) \iff \bigl((\forall k < \omega)(\mathbf{card}(Y_k) < \infty)\bigr)$$

where

$$Y_k = \bigl\{\alpha \in A\colon \mathbf{card}(U \cap V_\alpha) \leqslant k\bigr\}.$$

In the case $U \rhd (V_\alpha\colon \alpha \in A)$ we shall say that the set U *intersects uniformly* the family $(V_\alpha\colon \alpha \in A)$.

Let us state some easy properties of the introduced relations:

Lemma 5.6. (i) *If $U \sqsubseteq U'$ and U intersects uniformly the family $(V_\alpha\colon \alpha \in A)$ then the set U' intersects uniformly the family $(V_\alpha\colon \alpha \in A)$ as well*;

(ii) *If U intersects uniformly $(V_\alpha\colon \alpha \in A)$ and $V'_\alpha \supseteq V_\alpha - S$ for some finite set S and for all $\alpha \in A$ then U intersects uniformly $(V'_\alpha\colon \alpha \in A)$*;

(iii) $(U \rhd (V_\alpha\colon \alpha \in A))\&(A' \sqsubseteq A) \Longrightarrow (U \rhd (V_\alpha\colon \alpha \in A'))$.

Proof. (i) Let $\mathbf{card}(U - U') = m$ and let $k < \omega$. Then

$$\mathbf{card}(U' \cap V_\alpha) \geqslant \mathbf{card}(U \cap V_\alpha) - m$$

for all $\alpha \in A$, and

$$Y_k = \bigl\{\alpha \in A\colon \mathbf{card}(U' \cap V_\alpha) \leqslant k\bigr\} \subseteq \bigl\{\alpha \in A\colon \mathbf{card}(U \cap V_\alpha) \leqslant k + m\bigr\}.$$

The latter set is finite, and therefore Y_k is finite as well, and $U' \rhd (V_\alpha\colon \alpha \in A)$.

(ii) Let us denote $\mathbf{card}(S)$ by m. Then

$$\mathbf{card}(U \cap V'_\alpha) \geqslant \mathbf{card}(U \cap V_\alpha) - m$$

for all $\alpha \in A$, and the same reasoning as in (i) gives the result.

(iii) Let $\mathbf{card}(A' - A) = m$ and let $k < \omega$. Then

$$\begin{aligned}\mathbf{card}(Y_k) &= \mathbf{card}\big(\{\alpha \in A' \colon \mathbf{card}(U \cap V_\alpha) \leqslant k\}\big)\\ &\leqslant \mathbf{card}\big(\{\alpha \in A \colon \mathbf{card}(U \cap V_\alpha) \leqslant k\}\big) + m < \infty. \qquad \square\end{aligned}$$

Lemma 5.7. *Let $(V_i \colon i \in I)$ be a countable collection of subsets of a set Z. Let*

$$\big((g_i : V_i \to Y) \colon i \in I\big)$$

be a collection of almost agreeing mappings, and let for some $V \subseteq Z$

$$(\forall i \in I)(V_i \sqsubseteq V).$$

Then there exists a mapping

$$g : V \to Y$$

almost agreeing with all the mappings g_i.

Proof. Let $\varphi : [0, \omega\rangle \to I$ be a bijection, and let W_n be the following sets:

$$W_n = \left(V_{\varphi(n)} - \bigcup_{k=0}^{n-1} V_{\varphi(k)}\right) \cap V.$$

Let further

$$W_{-1} = \left(Z - \bigcup_{k=0}^{\infty} V_{\varphi(k)}\right) \cap V.$$

Clearly $(W_n \colon -1 \leqslant n < \infty)$ is a disjoint family of subsets of V, and

$$\bigcup_{k=-1}^{\infty} W_k = V.$$

Choose an element $y_0 \in Y$, and define $g : V \to Y$ as follows:

$$g(x) = \begin{cases} g_n(x) & \text{if } x \in W_n,\ n \geqslant 0,\\ y_0 & \text{if } x \in W_{-1}.\end{cases}$$

We need only to check that g is the desired “global” function, that is, g almost agrees with all the mappings g_i. Let $i \in I$, and choose n such that $i = \varphi(n)$. It follows that the set

$$\big\{x \in V_{\varphi(n)} \colon g(x) \neq g_n(x)\big\} \subseteq \bigcup_{k=0}^{n-1} \big\{x \in V_{\varphi(n)} \cap V_{\varphi(k)} \colon g_n(x) \neq g_k(x)\big\}$$

is finite, and g almost agrees with all g_i's. $\square$

Lemma 5.8. *Let $(A_n \colon n < \omega)$ and $(B_n \colon n < \omega)$ be two countable families of infinite subsets of Z such that for some subset T of Z and for all $m, n < \omega$,*

$$A_n \perp B_m \quad \textit{and} \quad \bigcup_{k=0}^{n} (A_k \cup B_k) \sqsubset T.$$

Then there exist subsets $A, B \subseteq T$ *such that*

$$A \perp B, \qquad A \cup B \sqsubset T,$$

$$A_m \sqsubseteq A \quad \textit{and} \quad B_m \sqsubseteq B \quad \textit{for every } m < \omega.$$

Proof. For each $n < \omega$ the set

$$T - \bigcup_{k=0}^{n} (A_n \cup B_n)$$

is infinite, and we can choose a subset

$$R_n \subseteq T - \bigcup_{k=0}^{n} (A_n \cup B_n)$$

satisfying $\mathbf{card}(R_n) = n$. Define a set

$$R = \bigcup_{n=0}^{\infty} R_n.$$

Clearly $R \subseteq T$, and R is countable. Remember that "countable" always means "infinite countable". We claim that

$$R \perp (A_m \cup B_n)$$

for all $m, n < \omega$. Indeed, there exists an $N < \omega$ such that $N \geqslant \max(m, n)$. Now

$$R_k \cap (A_m \cup B_n) = \emptyset$$

for all $k \geqslant N$, and

$$\bigl|R \cap (A_m \cup B_n)\bigr| \leqslant \left|\bigcup_{k=0}^{N-1} R_k\right| < \infty.$$

Let

$$V_{mn} = \begin{cases} A_n & \text{if } m = 0, \\ B_n & \text{if } m = 1, \end{cases}$$

and

$$g_{mn} : V_{mn} \to \{0, 1, 2, 3, 4\}$$

defined by $g_{mn}(x) = m$, $m = 0, 1$. Represent R as a disjoint union of three countable subsets:

$$R = S_2 \cup S_3 \cup S_4.$$

Define in addition a mapping

$$h_i : S_i \mapsto i \in \{2, 3, 4\} \subseteq \{0, 1, 2, 3, 4\}.$$

It follows from Lemma 5.7 that there exists a "global" function $g : T \to \{0, 1, 2, 3, 4\}$ almost agreeing with the three h_i and all g_{mn}. Clearly the sets

$$A := g^{-1}\bigl(\{0, 2\}\bigr), \qquad B := g^{-1}\bigl(\{1, 3\}\bigr)$$

are the desired ones. $\square$

Proposition 5.9 (Non-triviality lemma). *Let*

$$(V_\alpha\colon\ \alpha<\omega_1)$$

be a family of countable subsets of a (*not necessarily countable*) *set* V, *and let*

$$\mathcal{F}=(f_\alpha : V_\alpha \to G\colon\ \alpha<\omega_1)$$

be a family of almost agreeing functions. Assume in addition that there exist some $g\neq h$ *in* G *such that for any* $\alpha<\omega_1$

$$(f_\alpha)^{-1}(g) \rhd \bigl((f_\beta)^{-1}(h)\colon\ \beta<\alpha\bigr).$$

Then the family $\mathcal{F}$ *is not trivial.*

Proof. Suppose on the contrary that the family is trivial. Let

$$f : V \to G$$

be a global function almost agreeing with the family $\mathcal{F}$. The family

$$\bigl(\mathbf{card}\bigl(W_\alpha := \bigl\{x\in V_\alpha\colon\ f(x)\neq f_\alpha(x)\bigr\}\bigr)\colon\ \alpha<\omega_1\bigr)$$

has uncountable number of members, and therefore there exists an uncountable subfamily of identical integers:

$$\exists N\bigl(B_N := \bigl\{\alpha\colon\ \mathbf{card}(W_\alpha)=N\bigr\} \text{ is uncountable}\bigr).$$

The subset B_N possesses an infinite monotone subsequence

$$(\beta_0<\beta_1<\beta_2<\cdots,\ \beta_i\in B_N).$$

Since B_N is uncountable, there exists an $\alpha\in B_N$ such that

$$\alpha \geqslant \lim_{n\to\infty}\beta_n.$$

Let

$$U=(f_\alpha)^{-1}(g) \quad \text{and} \quad V_\beta=(f_\beta)^{-1}(h).$$

It follows that

$$U\cap V_{\beta i}\subseteq W_\alpha\cup W_{\beta_i},$$

and therefore,

$$\mathbf{card}(U\cap V_{\beta i})\leqslant \mathbf{card}(W_\alpha)+\mathbf{card}(W_{\beta_i})=2N.$$

We have found an infinite number of indexes with $\mathbf{card}(U\cap V_{\beta i})\leqslant 2N$. This contradicts the condition

$$(f_\alpha)^{-1}(g) \rhd \bigl((f_\beta)^{-1}(h)\colon\ \beta<\alpha\bigr). \qquad \square$$

Proposition 5.10 (Induction lemma). *Let* Z *be a set, let* $T\subseteq Z$, *let* $\alpha<\omega_1$ *and let*

$$\mathcal{U}=(U_\beta\colon\ \beta<\alpha), \qquad \mathcal{V}=(V_\beta\colon\ \beta<\alpha)$$

and

$$\mathcal{W} = (W_\beta\colon\ \beta < \alpha)$$

be families of countable subsets of Z, *indexed by the set* $[0, \alpha\rangle$ *and such that* $\mathcal{U} \perp \mathcal{W}$,

$$(\forall\beta)\big((\beta < \alpha) \Rightarrow (V_\beta \subseteq W_\beta)\&(U_\beta \cup W_\beta \sqsubset T)\big),$$

and

$$(\forall\beta_0, \beta_1)\big((\beta_0 < \beta_1 < \alpha) \Rightarrow (U_{\beta_0} \sqsubset U_{\beta_1})\&(V_{\beta_0} \sqsubset V_{\beta_1})\&(W_{\beta_0} \sqsubset W_{\beta_1})\big).$$

Assume in addition that for each $\beta < \alpha$ *the set* U_β *intersects uniformly the family*

$$(V_\gamma\colon\ \gamma < \beta).$$

Then there exists a countable subset $U \subseteq Z$ *such that* $U \sqsubset T$, $U \perp \mathcal{W}$, $U \rhd \mathcal{V}$, *and*

$$(\forall\beta)\big((\beta < \alpha) \Rightarrow (U_\beta \sqsubset U)\big).$$

Proof. *Case* 1. $\alpha = \alpha_0 + 1$. Then the set

$$S = T - (U_{\alpha_0} \cup W_{\alpha_0})$$

is infinite, and we can choose countable subsets

$$R_0, \qquad R_1 \subseteq S, \qquad R_0 \cap R_1 = \emptyset.$$

Set $U = U_{\alpha_0} \cup R_0$. It follows from Lemma 5.6 that U satisfies the conditions of the Induction lemma.

Case 2. α is a limit ordinal,

$$\alpha = \lim_{n\to\omega} \alpha_n$$

where $(\alpha_n\colon\ n < \omega)$ is a strictly increasing sequence of ordinals. Apply Lemma 5.8 to the families

$$(U_\beta\colon\ \beta < \alpha)$$

and

$$(W_\beta\colon\ \beta < \alpha).$$

There exist sets

$$U, W \subseteq T, \qquad U \perp W, \qquad U \cup W \sqsubset T$$

such that

$$(\forall\beta < \alpha)\big((U_\beta \sqsubset U)\&(W_\beta \sqsubset W)\big).$$

Let

$$R \subseteq T - (U \cup W)$$

be a countable subset. We shall construct inductively a sequence

$$U - R =: S =: S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq T - R \sqsubset T$$

of subsets of $T - R$ with the property

$$(\forall k < \omega)\big[(S_k \perp W)\&\big(S_k \rhd (V_{\alpha'}\colon \alpha' \in B_k)\big)\big]$$

where

$$B_k = \big\{\alpha' \in [0, \alpha\rangle\colon \mathbf{card}(S \cap V_{\alpha'}) \leqslant k\big\}.$$

If $B = B_k - B_{k-1}$ is finite, then we set $S_{k+1} = S_k$. Otherwise B is cofinal in $[0, \alpha\rangle$ and has the ordinal type ω. Indeed,

$$S \sqsupseteq U_\beta \rhd \big(V_{\alpha'}\colon \alpha' \in B \cap [0, \beta\rangle\big)$$

for every $\beta < \alpha$, and therefore $B \cap [0, \beta\rangle$ is finite. Let $b : [0, \omega\rangle \to B$ be a monotone bijection, let

$$c : [0, \omega\rangle \to [0, \alpha\rangle$$

be some bijection, and let

$$B'_n = \big\{c(k) : (k \leqslant n)\&\big(c(k) < b(n)\big)\big\}$$

for $n < \omega$. Now the sets

$$V'_n = \Big(V_{b(n)} - \bigcup\{W_b\colon b \in B'_n\}\Big)$$

are infinite for all $n < \omega$, because $V_{b(n)} - W_b$ are infinite for $b < b(n)$. The sets

$$T_n = (T - R) \cap V'_n$$

are also infinite, because $V_b \sqsubseteq T - R$ for all $b < \alpha$. We shall choose now a subset P_n in every set T_n such that $\mathbf{card}(P_n) = n$, and let P be the union

$$P = \bigcup\{P_n\colon n < \omega\}.$$

Clearly,

$$P \rhd (V_{\alpha'}\colon \alpha' \in B),$$

since for any $b(n) \in B$

$$\mathbf{card}(P \cap V_{b(n)}) \geqslant \mathbf{card}(P_n) = n$$

and

$$\lim_{n\to\infty} \mathbf{card}(P \cap V_{b(n)}) = \infty.$$

Moreover

$$P \perp (W_{\alpha'}\colon \alpha' < \alpha).$$

Indeed, for any $n < \omega$ there exists an $m < \omega$ with $b(m) > c(n)$. Let $k = \max(m, n)$. $R_l \cap W_{c(n)} = \emptyset$, for all $l \geqslant k$, because $R_l \subseteq V'_n$. Thus,

$$\mathbf{card}(R \cap W_{c(n)}) < \infty,$$

as desired. We set simply $S_{k+1} = S_k \cup R$. Apply again Lemma 5.8 to the families

$$S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$$

and

$$(W_\beta\colon\ \beta < \alpha),$$

and obtain a set $U \subseteq T - R$ with all the desired properties. $\square$

Proposition 5.11. *Let $(V_\alpha\colon\ \alpha < \omega_1)$ be an ω_1-indexed family of infinite subsets of Z such that*

$$(\forall \alpha, \beta)\big((\alpha < \beta < \omega_1) \Rightarrow V_\alpha \sqsubset V_\beta\big).$$

Then there exists a family

$$\big(\big(\sigma_\alpha : V_\alpha \to [0,\alpha] \subseteq [0,\omega_1\rangle\big)\colon\ \alpha < \omega_1\big)$$

of mappings satisfying the following conditions:

(a) *The family is almost agreeing*;
(b) $(\forall \alpha < \omega_1)(\forall \beta \leqslant \alpha)(\mathbf{card}(\sigma_\alpha^{-1}\{\beta\}) = \infty)$;
(c) $(\forall \beta < \omega_1)(\forall \alpha_1 \geqslant \beta)(\forall \alpha_2 > \alpha_1)(\sigma_{\alpha_1}^{-1}(\beta) \sqsubset \sigma_{\alpha_2}^{-1}(\beta))$;
(d) $(\forall \alpha < \omega_1)(\forall \beta, 0 < \beta \leqslant \alpha)(\sigma_\alpha^{-1}(0) \rhd (\sigma_{\alpha'}^{-1}(\beta)\colon\ \beta \leqslant \alpha' < \alpha))$.

Proof. It suffices to prove that there exists a family of subsets

$$(A_{\beta\alpha}\colon\ \beta \leqslant \alpha < \omega_1)$$

such that

$$A_{\beta\alpha} \cap A_{\gamma\alpha} = \emptyset \quad \text{for all } \beta \neq \gamma,$$

$$\bigcup_{\beta \leqslant \alpha} A_{\beta\alpha} = V_\alpha,$$

and

(a) $\bigcup_{\beta \neq \gamma}(A_{\beta\alpha} \cap A_{\gamma\alpha'})$ is finite for all $\alpha' < \alpha < \omega_1$;
(b) All the sets $A_{\beta\alpha}$ are infinite;
(c) $A_{\beta\alpha_1} \sqsubset A_{\beta\alpha_2}$ for all triples $\beta \leqslant \alpha_1 < \alpha_2$;
(d) The set $A_{0\alpha}$ intersects uniformly the family $(A_{\beta\alpha'}\colon\ \alpha' \in [\beta,\alpha\rangle)$ for all pairs $0 < \beta < \alpha < \omega_1$.

Then the functions

$$\sigma_\alpha : V_\alpha \to [0,\alpha],$$

defined by

$$\sigma_\alpha|_{A_{\beta\alpha}} = \beta,$$

for $\beta \leqslant \alpha$, satisfy the requirements of Proposition 5.11.

We shall construct the family $A_{\beta\alpha}$ using transfinite induction relative to α.

Step $\alpha = 0$. Set $A_{00} = V_0$.

Step $(<\alpha) \Longrightarrow \alpha$. Let sets $A_{\beta\alpha'}$ satisfy (a)–(d) for all $i<\omega$, $\beta<\alpha'<\alpha$.
Choose β, $0<\beta<\alpha$. Set $T=V_\alpha$,

$$\mathcal{U}=(A_{0\gamma}\colon\ \gamma<\alpha<\omega_1), \qquad \mathcal{V}=(A_{\beta\gamma}\colon\ \beta\leqslant\gamma<\alpha<\omega_1)$$

and

$$\mathcal{W}=(W_\gamma:=V_\gamma-A_{0\gamma}\colon\ \gamma<\alpha<\omega_1),$$

and apply the Induction lemma. The lemma gives us a subset (which will be denoted by S_β) of V_α with the properties:

$$(\forall\gamma<\alpha)(A_{0\gamma}\sqsubset S_\beta);$$

$$S_\beta\perp(V_\gamma-A_{0\gamma}\colon\ \gamma<\alpha);$$

$$S_\beta\rhd(A_{\beta\gamma}\colon\ \beta\leqslant\gamma<\alpha<\omega_1).$$

Now apply the Induction lemma to *all* β, $0<\beta<\alpha$, and obtain the sets S_β with the above properties. Apply now Lemma 5.8 to the (countable!) families $(S_\beta\colon\ 0<\beta<\alpha)$ and $\mathcal{W}$, and get subsets $A_{0\alpha}$ and W of V_α with the following properties:

$$A_{0\alpha}\perp W;$$

$$(\forall\beta<\alpha)\big((A_{0\beta}\sqsubset A_{0\alpha})(W_\beta\sqsubset W)\big);$$

$$A_{0\alpha}\cup W\sqsubset V_\alpha;$$

$$\big(\forall\gamma\in[1,\alpha\rangle\big)A_{0\alpha}\rhd(A_{\gamma\beta}\colon\ \gamma\leqslant\beta<\alpha).$$

Let

$$Q=V_\alpha-(A_{0\alpha}\cup W),$$

which is a (infinite!) countable set. Now consider following family of sets:

$$\{A_{0\alpha}\}\cup\{Q\}\cup(V_\beta\colon\ \beta<\alpha)$$

and following family of mappings:

$$g_0:A_{0\alpha}\mapsto 0\in[0,\alpha],$$

$$g_\alpha:Q\mapsto\alpha\in[0,\alpha],$$

$$\sigma_\beta:V_\beta\mapsto 0\in[0,\alpha].$$

The family of mappings is clearly almost agreeing, and we can apply Lemma 5.7 and get the desired mapping

$$\sigma_\alpha:V_\alpha\to[0,\alpha]. \qquad \square$$

Now we are ready to prove Theorems 3 and 4. Using the condition $d=\aleph_1$ one can easily construct a family

$$J=\big(\lambda_\alpha\in[0,\omega\rangle^{[0,\omega\rangle}\colon\ \alpha<\omega_1\big),$$

cofinal in $\Lambda' = ([0,\omega\rangle^{[0,\omega\rangle}, \ll)$, such that $\lambda_\alpha \ll \lambda_\beta$ for all $\alpha < \beta < \omega_1$. We claim in addition that the inequality $\lambda_\alpha \ll \lambda_\beta$ is *strict* for all pairs $\alpha < \beta$. Let us define for each $\alpha < \omega_1$

$$V_\alpha = U_{\lambda_\alpha} = \bigl\{(i,j) \in [0,\omega\rangle \times [0,\omega\rangle\colon\ j \leqslant \lambda_\alpha(i)\bigr\}.$$

The family $(V_\alpha\colon\ \alpha < \omega_1)$ satisfies the conditions of Proposition 5.11 for $Z = [0,\omega\rangle \times [0,\omega\rangle$, and therefore there exists a family

$$\bigl(\bigl(\sigma_\alpha : V_\alpha \to [0,\alpha] \subseteq [0,\omega_1\rangle\bigr)\colon\ \alpha < \omega_1\bigr)$$

satisfying (a)–(d) from Proposition 5.11. Let $\varphi : [1,\omega\rangle \to G$ be a mapping. We shall define a family of functions

$$\Gamma_\varphi = (g_{\varphi\alpha} : V_\alpha \to G)$$

where

$$g_{\varphi\alpha}(x) = \begin{cases} \varphi(\sigma_\alpha(x)) & \text{if } \sigma_\alpha(x) \neq 0, \\ 0 & \text{if } \sigma_\alpha(x) = 0. \end{cases}$$

The family Γ_φ corresponds to an element from $\mathbf{lim}(\mathbf{R}(G))$, and therefore to an element

$$h_\varphi \in \mathbf{coker}\bigl(G^{[0,\omega\rangle\times[0,\omega\rangle} \to \mathbf{lim}\bigl(\mathbf{R}(G)\bigr)\bigr) \approx \mathbf{lim}^1\mathbf{P}^\omega(G) \approx c_1(G)$$

(see Corollary 5.3). The mapping $\varphi \mapsto h_\varphi$ from $G^{[1,\omega\rangle}$ to $c_1(G)$ is clearly a group homomorphism, and we shall prove that it is a monomorphism. Suppose on the contrary that there exists a function

$$0 \neq \varphi : [1,\omega_1\rangle \to G$$

such that $h_\varphi = 0$. Let $h = \varphi(\alpha) \neq 0$, and $g = 0$.

Apply the non-triviality lemma (Proposition 5.9) to the family

$$\mathcal{F} = (f_\alpha = \varphi \circ \sigma_\alpha : V_\alpha \to G\colon\ \alpha < \omega_1)$$

of almost agreeing functions. The non-triviality lemma implies that the family $\mathcal{F}$ is not trivial. This contradiction shows that the homomorphism $G^{[1,\omega_1\rangle} \to c_1(G)$ is injective, and thus

$$\mathbf{card}\bigl(c_1(G)\bigr) \geqslant \bigl(\mathbf{card}(G)\bigr)^{\aleph_1}.$$

At the same time

$$\bigl|c_1(G)\bigr| \leqslant \bigl|\lim \mathbf{R}(G)\bigr| \leqslant |G|^{\aleph\times\aleph_1} = |G|^{\aleph_1}$$

and therefore $|c_1(G)| = |G|^{\aleph_1}$ as desired.

Theorems 3 and 4 have been proven.

6. Higher limits are not additive

Given a group G, let $\mathbf{A}(G)$ be the following pro-group:

$$\mathbf{A}(G) = \Bigl(\mathbf{A}(G)_\alpha = \bigoplus_{\beta\in[\alpha,\omega_1\rangle} G\colon\ \alpha < \omega_1\Bigr)$$

with the natural embeddings

$$p_{\alpha_2\alpha_1}: \mathbf{A}(G)_{\alpha_1} \hookrightarrow \mathbf{A}(G)_{\alpha_2}, \quad \alpha_2 < \alpha_1,$$

induced by inclusions $[\alpha_1, \omega_1\rangle \subseteq [\alpha_2, \omega_1\rangle$.

Consider the direct sum

$$\mathbf{A}^\omega(G) = \bigoplus_{i<\omega} \mathbf{A}(G)$$

in the category **pro-AB**. Let

$$\Lambda := [0, \omega_1\rangle^{[0,\omega\rangle} := \bigl\{\lambda : [0, \omega\rangle \to [0, \omega_1\rangle\bigr\}$$

with the componentwise ordering. Then the pro-group $\mathbf{A}^\omega(G)$ can be described as follows (see Lemma 4.7):

$$\mathbf{A}^\omega(G) = \left(\mathbf{A}^\omega(G)_\lambda = \bigoplus_{i<\omega}\left(\bigoplus_{\beta \geqslant \lambda(i)} G\right):\ \lambda \in \Lambda\right)$$

with the evident natural embeddings. However, the constant functions $\lambda(i) \equiv \alpha < \omega_1$ form a cofinal subset of Λ. Choose a monotone embedding (automatically cofinal)

$$\gamma : [0, \omega_1\rangle \to [0, \omega_1\rangle$$

such that for all $\alpha_0 < \alpha_1 < \omega_1$ the set $[\gamma(\alpha_0), \gamma(\alpha_1)]$ is infinite (countable!). Assume for convenience that $[0, \gamma(0)]$ is also infinite. Now the pro-group $\mathbf{A}^\omega(\mathbf{G})$ admits the following description up to isomorphism:

$$\mathbf{A}^\omega(G) = \left(\mathbf{A}^\omega(G)_\alpha = \bigoplus_{i<\omega}\left(\bigoplus_{\beta \geqslant \gamma(\alpha)} G\right):\ \alpha < \omega_1\right)$$

with the natural *embeddings*

$$p_{\alpha_2\alpha_1}: \mathbf{A}^\omega(G)_{\alpha_1} \hookrightarrow \mathbf{A}^\omega(G)_{\alpha_2}, \quad \alpha_2 < \alpha_1.$$

Let $\mathbf{B}(G)$ be the constant pro-group:

$$\mathbf{B}(G) = \left(\mathbf{B}(G)_\alpha = \bigoplus_{\beta<\omega_1} G:\ \alpha < \omega_1\right)$$

and let $\mathbf{C}(G)$ be the factorgroup $\mathbf{B}(G)/\mathbf{A}(G)$. Define analogously

$$\mathbf{B}^\omega(G) = \bigoplus_{i<\omega} \mathbf{B}(G), \qquad \mathbf{C}^\omega(G) = \bigoplus_{i<\omega} \mathbf{C}(G).$$

It is easy to see that up to isomorphism

$$\mathbf{C}(G) = \left(\mathbf{C}(G)_\alpha = \bigoplus_{\beta<\alpha} G:\ \alpha < \omega_1\right) \approx \left(\mathbf{C}(G)_\alpha = \bigoplus_{\beta<\gamma(\alpha)} G:\ \alpha < \omega_1\right),$$

$$\begin{aligned}\mathbf{C}^\omega(G) &= \left(\mathbf{C}^\omega(G)_\alpha = \bigoplus_{i<\omega}\bigoplus_{\beta<\alpha} G:\ \alpha < \omega_1\right)\\ &\approx \left(\mathbf{C}^\omega(G)_\alpha = \bigoplus_{i<\omega}\left(\bigoplus_{\beta<\gamma(\alpha)} G\right):\ \alpha < \omega_1\right),\end{aligned}$$

with the natural *projections*, the pro-group $\mathbf{B}^{\omega}(G)$ is a constant pro-group:

$$\mathbf{B}^{\omega}(G) = \left(\mathbf{B}^{\omega}(G)_{\alpha} = \bigoplus_{i<\omega}\bigoplus_{\beta<\omega_1} G\colon \alpha < \omega_1\right),$$

and $\mathbf{C}^{\omega}(G)$ is isomorphic to the factorgroup $\mathbf{B}^{\omega}(G)/\mathbf{A}^{\omega}(G)$.

Proposition 6.1. (a)

$$\mathbf{lim}^n \mathbf{B}(G) = \mathbf{lim}^n \mathbf{B}^{\omega}(G) = 0$$

for $n \geqslant 1$;

(b)

$$\mathbf{lim}^n \mathbf{A}(G) = \mathbf{lim}^n \mathbf{A}^{\omega}(G) = \mathbf{lim}^n \mathbf{C}(G) = \mathbf{lim}^n \mathbf{C}^{\omega}(G) = 0$$

for $n \geqslant 3$;

(c)

$$\begin{aligned}\mathbf{lim}^2 \mathbf{C}(G) &= \mathbf{lim}^2 \mathbf{C}^{\omega}(G) = \mathbf{lim}^1 \mathbf{A}(G)\\ &= \mathbf{lim}^1 \mathbf{A}^{\omega}(G) = \mathbf{lim}\,\mathbf{A}(G) = \mathbf{lim}\,\mathbf{A}^{\omega}(G) = 0;\end{aligned}$$

(d)

$$\mathbf{lim}^2 \mathbf{A}(G) \approx \mathbf{lim}^1 \mathbf{C}(G) \neq 0; \qquad \mathbf{lim}^2 \mathbf{A}^{\omega}(G) \approx \mathbf{lim}^1 \mathbf{C}^{\omega}(G) \neq 0.$$

Proof. (a) Both pro-groups $\mathbf{B}(G)$ and $\mathbf{B}^{\omega}(G)$ are constant;

(b) The index set ω_1 has cardinality $\aleph_1$, and therefore $\mathbf{lim}^n = 0$ for $n \geqslant 3$ (see [15]);

(c) The long exact sequences arising from the two short exact sequences

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0,$$

$$0 \to \mathbf{A}^{\omega} \to \mathbf{B}^{\omega} \to \mathbf{C}^{\omega} \to 0$$

give $\mathbf{lim}^2 \mathbf{C}(G) = \mathbf{lim}^3 \mathbf{A}(G) = 0$ and $\mathbf{lim}^2 \mathbf{C}^{\omega}(G) = \mathbf{lim}^3 \mathbf{A}^{\omega}(G) = 0$.

(d) From the above exact sequences it follows also that $\mathbf{lim}^2 \mathbf{A}(G) \approx \mathbf{lim}^1 \mathbf{C}(G)$ and $\mathbf{lim}^2 \mathbf{A}^{\omega}(G) \approx \mathbf{lim}^1 \mathbf{C}^{\omega}(G)$. That those groups are non-trivial, will be proven in Section 6.1.

Moreover,

$$\mathbf{lim}^1 \mathbf{A}(G) \approx \mathbf{coker}\left(\bigoplus_{i<\omega_1} G = \mathbf{lim}\,\mathbf{B}(G) \to \mathbf{lim}\big(\mathbf{C}(G)\big) = \bigoplus_{i\in\omega_1} G\right) = 0,$$

$$\mathbf{lim}\,\mathbf{A}(G) \approx \mathbf{ker}\left(\bigoplus_{i\in\omega_1} G = \mathbf{lim}\,\mathbf{B}(G) \to \mathbf{lim}\,\mathbf{C}(G) = \bigoplus_{i\in\omega_1} G\right) = 0,$$

$$\mathbf{lim}^1 \mathbf{A}^{\omega}(G) \approx \mathbf{coker}\left(\bigoplus_{i\in\omega}\bigoplus_{i\in\omega_1} G = \mathbf{lim}\,\mathbf{B}^{\omega}(G) \to \mathbf{lim}\,\mathbf{C}(G) = \bigoplus_{i\in\omega}\bigoplus_{i\in\omega_1} G\right) = 0,$$

$$\mathbf{lim}\,\mathbf{A}^{\omega}(G) \approx \mathbf{ker}\left(\bigoplus_{i\in\omega}\bigoplus_{i\in\omega_1} G = \mathbf{lim}\,\mathbf{B}^{\omega}(G) \to \mathbf{lim}\,\mathbf{C}^{\omega}(G) = \bigoplus_{i\in\omega}\bigoplus_{i\in\omega_1} G\right) = 0. \qquad \square$$

6.1. Proof of Theorem 1

Let us define the following pro-groups:

$$\mathbf{D}(G) = \left(\mathbf{D}(G)_\alpha = \prod_{\beta<\alpha} G\colon\ \alpha < \omega_1\right),$$

$$\mathbf{D}^\omega(G) = \left(\mathbf{D}^\omega(G)_\alpha = \prod_{i\in\omega}\prod_{\beta<\alpha} G\colon\ \alpha < \omega_1\right)$$

with natural projections as bonding homomorphisms. The pro-groups $\mathbf{C}(G)$ and $\mathbf{C}^\omega(G)$ are naturally included in $\mathbf{D}(G)$ and $\mathbf{D}^\omega(G)$ respectively, and let us define

$$\mathbf{E}(G) = \mathbf{D}(G)/\mathbf{C}(G); \qquad \mathbf{E}^\omega(G) = \mathbf{D}^\omega(G)/\mathbf{C}^\omega(G).$$

Proposition 6.2. (a)

$$\mathbf{lim}^1\mathbf{C}(G) \approx \mathbf{coker}\big(G^{\omega_1} = \mathbf{lim}\,\mathbf{D}(G) \to \mathbf{lim}\,\mathbf{E}(G)\big);$$

(b)

$$\mathbf{lim}^1\mathbf{C}^\omega(G) \approx \mathbf{coker}\big(G^{\omega\times\omega_1} = \mathbf{lim}\,\mathbf{D}^\omega(G) \to \mathbf{lim}\,\mathbf{E}^\omega(G)\big).$$

Proof. (a) Set $S = \{0\}$, $\alpha = \omega_1$, and apply Proposition 5.1:

$$\mathbf{lim}^1\mathbf{P}_\alpha(G)^S \approx \mathbf{coker}\big(G^{S\times[0,\alpha\rangle} \to \mathbf{lim}\big(\mathbf{R}^S_\alpha(G)\big)\big)$$

where

$$\mathbf{Q}^S_\alpha(G) \approx \mathbf{D}(G), \quad \mathbf{R}^S_\alpha(G) \approx \mathbf{E}(G), \quad \mathbf{P}_\alpha(G)^S \approx \mathbf{C}(G).$$

(b) Set $S = [0,\omega\rangle$, $\alpha = \omega_1$, and again apply Proposition 5.1:

$$\mathbf{lim}^1\mathbf{P}_\alpha(G)^S \approx \mathbf{coker}\big(G^{S\times[0,\alpha\rangle} \to \mathbf{lim}\big(\mathbf{R}^S_\alpha(G)\big)\big)$$

where

$$\mathbf{Q}^S_\alpha(G) \approx \mathbf{D}^\omega(G), \quad \mathbf{R}^S_\alpha(G) \approx \mathbf{E}^\omega(G), \quad \mathbf{P}_\alpha(G)^S \approx \mathbf{C}^\omega(G). \qquad \square$$

Using the above proposition and the monotone mapping

$$\gamma : [0,\omega_1\rangle \to [0,\omega_1\rangle,$$

we can describe the elements of $\mathbf{lim}^1\mathbf{C}(G)$ (respectively $\mathbf{lim}^1\mathbf{C}^\omega(G)$), analogously to Section 5 as equivalence classes of families

$$\big(f_\alpha : \{\beta\colon\ \beta < \gamma(\alpha)\} \to G\colon\ \alpha < \omega_1\big),$$

respectively

$$\big(f_\alpha : [0,\omega\rangle \times \{\beta\colon\ \beta < \gamma(\alpha)\} \to G\colon\ \alpha < \omega_1\big)$$

of "almost agreeing" functions. The latter means that $f_\alpha = f_\beta$ on the intersection of their domains except for a finite number of points. Two such collections (f_α) and (g_α) are equivalent iff there exists a "global" function

$$h : [0,\omega_1\rangle \to G \ \big(\text{respectively } h : [0,\omega\rangle \times [0,\omega_1\rangle \to G\big)$$

such that

$$f_\alpha - g_\alpha \equiv h|_{D_\alpha}$$

($\equiv$ means "almost agree") where D_α is the definition domain for f_α and g_α.

In Section 6.2 we will construct a certain element $\xi \in \mathbf{lim}^1\mathbf{C}^\omega(G)$, represented by a family

$$\bigl(f_\alpha : [0,\omega\rangle \times \bigl[0,\gamma(\alpha)\bigr\rangle \to G\colon\ \alpha < \omega_1\bigr)$$

of almost agreeing functions. Choose a non-zero element $g \in G$. The functions f_α will have values in the set $\{0, g\}$, and therefore, they will give rise to a family of subsets

$$\mathcal{U} := \bigl(U_{i\alpha} \subseteq V_\alpha = [0,\omega\rangle \times \bigl[0,\gamma(\alpha)\bigr\rangle\colon\ i \in \{0,1\},\ \alpha < \omega_1\bigr)$$

where

$$U_{0\alpha} = (f_\alpha)^{-1}(0), \qquad U_{1\alpha} = (f_\alpha)^{-1}(g).$$

The sets $U_{i\alpha}$ will have the following properties:

$$U_{0\alpha} \cup U_{1\alpha} = V_\alpha, \qquad U_{0\alpha} \cap U_{1\alpha} = \emptyset;$$

$$(\forall\beta)\bigl((\beta < \alpha) \Rightarrow (V_\beta \sqsubset V_\alpha)\bigr),$$

and

$$(\forall i < \omega)(\forall \beta_0, \beta_1)\bigl((\beta_0 < \beta_1 < \alpha)$$
$$\Rightarrow (U_{0\beta_0} \cap L_i \sqsubset U_{0\beta_1} \cap L_i)\&(U_{1\beta_0} \cap L_i \sqsubset U_{1\beta_1} \cap L_i)\bigr),$$
$$(\forall\alpha < \omega_1)(\forall i < \omega)\bigl(U_{0\alpha} \rhd (U_{1\beta} \cap L_i\colon\ \beta < \alpha)\bigr)$$

where

$$L_i = \{i\} \times [0,\omega_1\rangle$$

(see the definitions from Section 5.1).

Proposition 6.3. *Let*

$$(p_i)^* : \mathbf{lim}^1\mathbf{C}^\omega(G) \to \mathbf{lim}^1\mathbf{C}(G), \quad i < \omega,$$

be the homomorphism induced by the ith natural projection

$$p_i : \mathbf{C}^\omega(G) = \bigoplus_{i<\omega} \mathbf{C}(G) \to \mathbf{C}(G).$$

Then $(p_i)^*(\xi) \neq 0$ for all $i < \omega$.

Proof. The element

$$(p_i)^*(\xi) \in \mathbf{lim}^1\mathbf{C}(G)$$

is represented by the family

$$\mathcal{F}_i := \bigl(f_{i\alpha} = f_\alpha \circ j_i : \bigl[0,\gamma(\alpha)\bigr\rangle \to G\colon\ \alpha < \omega_1\bigr)$$

where j_i is the inclusion of $[0, \gamma(\alpha)\rangle$ into V_α at level i:

$$j_i(\beta) := i \times \beta.$$

The properties of the family $\mathcal{U}$ imply that the family $\mathcal{F}_i$ satisfies the conditions of the non-triviality lemma (Proposition 5.9), and therefore the family $\mathcal{F}_i$ is not trivial, and

$$0 \neq (p_i)^*(\xi) \in \mathbf{lim}^1 \mathbf{C}(G). \qquad \square$$

Corollary 6.4. $\mathbf{lim}^1 \mathbf{C}(G) \neq 0$. $\square$

Corollary 6.5. ξ does not lie *in the image of*

$$\varphi : \bigoplus_{i<\omega} \mathbf{lim}^1 \mathbf{C}(G) \to \mathbf{lim}^1 \left(\bigoplus_{i<\omega} \mathbf{C}(G) \right),$$

and therefore φ *is* not *an* isomorphism.

Proof. If, on the contrary, ξ *does* lie in the image of φ, then only a *finite* number of projections $(p_i)^*(\xi)$ are non-zero, which is a contradiction. $\square$

Theorem 1 has been proved!

6.2. *The construction of ξ by transfinite induction on α*

Step $\alpha = 0$. Represent the countable set $[0, \gamma(\alpha)\rangle$ as a disjoint union of two countable sets

$$\big[0, \gamma(\alpha)\big\rangle = P \cup Q.$$

Define

$$U_{0\alpha} := [0, \omega\rangle \times P; \qquad U_{1\alpha} := [0, \omega\rangle \times Q,$$

and we are done.

Step $(< \alpha) \Longrightarrow \alpha$.

Case $\alpha = \alpha_0 + 1$. Represent the countable set $[\gamma(\alpha_0), \gamma(\alpha)\rangle$ as a disjoint union of two countable sets

$$\big[\gamma(\alpha_0), \gamma(\alpha)\big\rangle = P \cup Q.$$

Define

$$U_{0\alpha} := U_{0\alpha_0} \cup [0, \omega\rangle \times P, \qquad U_{1\alpha} := U_{1\alpha_0} \cup [0, \omega\rangle \times Q,$$

and we are done.

Case α is a limit ordinal.

Fix $i < \omega$, set $T = Z = V_\alpha$,

$$\mathcal{U} = (U_\beta := U_{0\beta}\colon\ \beta < \alpha), \qquad \mathcal{V}_i = (V_\beta := U_{1\beta} \cap L_i\colon\ \beta < \alpha)$$

and

$$\mathcal{W} = (W_\beta := U_{1\beta}\colon\ \beta < \alpha).$$

The conditions of the Induction lemma (Proposition 5.10) are clearly satisfied. Therefore, there exists a countable subset $S_i \subseteq V_\alpha$ such that $S_i \sqsubset V_\alpha$, $S_i \perp \mathcal{W}$, $S_i \rhd \mathcal{V}_i$, and

$$(\forall\beta)\big((\beta<\alpha) \Rightarrow (U_{0\beta} \sqsubset S_i)\big).$$

Let now i vary, and apply Lemma 5.8 to the (countable!) families $(S_i\colon\ i<\omega)$ and $(W_\beta\colon\ \beta<\alpha)$ of subsets of

$$T := Z := V_\alpha,$$

and get sets $A, B \subseteq V_\alpha$, such that, for all i,

$$S_i \sqsubset A, \qquad A \subseteq V_\alpha, \qquad A \subseteq A \cup B \sqsubset V_\alpha.$$

The set $U_{0\alpha} := A$ has all the desired properties. Set finally $U_{1\alpha} := V_\alpha - U_{0\alpha}$, and the induction step is done.

7. Strong homology is not additive

Consider the space

$$X_m = X(m, 0, \omega_1), \quad m \geqslant 0,$$

from [10]. As a set, X_m is a wedge

$$X_m = \bigvee_{\alpha<\omega_1} B^m$$

of m-dimensional balls equipped with some special paracompact topology. Let

$$X_m^\omega = \coprod_{i<\omega} X_m$$

be the topological coproduct of countably many copies of X_m. In [10], a polyhedral resolution for X_m was constructed, and the pro-homology groups were calculated. Those formulae for **pro**-$H_n(X_m, \mathbb{Z})$ can be easily re-established for **pro**-$H_n(X_m, G)$:

Proposition 7.1 (Compare [10, Theorems 3 and 6]).

$$\mathbf{pro}\text{-}H_n(X_m, G) = \begin{cases} G & \textit{if } n = 0 \neq m, \\ \mathbf{A}(G) & \textit{if } n = m, \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary 7.2.

$$\mathbf{pro}\text{-}H_n(X_m^\omega, G) = \begin{cases} G & \textit{if } n = 0 \neq m, \\ \mathbf{A}^\omega(G) & \textit{if } n = m, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Use additivity of pro-homology. □

The following proposition gives full description of strong homology of X_m and X_m^ω:

Proposition 7.3. (a)

$$\overline{H}_p(X_m,G)=\begin{cases} G & \textit{if } p=0 \textit{ and } m\neq 2,\\ G\oplus \mathbf{lim}^2\mathbf{A}(G)\approx G\oplus \mathbf{lim}^1\mathbf{C}(G) & \textit{if } p=0 \textit{ and } m=2,\\ \mathbf{lim}^2\mathbf{A}(G)\approx \mathbf{lim}^1\mathbf{C}(G) & \textit{if } p=m-2 \textit{ and } m\neq 2,\\ 0 & \textit{otherwise.}\end{cases}$$

(b)

$$\overline{H}_p(X_m^\omega,G)=\begin{cases} \bigoplus_{i\in\omega} G & \textit{if } p=0 \textit{ and } m\neq 2,\\ (\bigoplus_{i\in\omega} G)\oplus \mathbf{lim}^2\mathbf{A}^\omega(G) & \\ \quad\approx (\bigoplus_{i\in\omega} G)\oplus \mathbf{lim}^1\mathbf{C}^\omega(G) & \textit{if } p=0 \textit{ and } m=2,\\ \mathbf{lim}^2\mathbf{A}^\omega(G)\approx \mathbf{lim}^1\mathbf{C}^\omega(G) & \textit{if } p=m-2 \textit{ and } m\neq 2,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. (a) Consider the spectral sequence from Theorem 4.4:

$$E_2^{st}=\lim{}^s\,\mathbf{pro}\text{-}H_{-t}(X,G)\quad\Longrightarrow\quad \overline{H}_{-s-t}(X,G).$$

In the case $X=X_m$ the sequence degenerates:

$$E_2^{st}=\begin{cases} \bigoplus_{i\in\omega} G & \text{if } s=t=0,\\ \mathbf{lim}^2\mathbf{A}(G)\approx \mathbf{lim}^1\mathbf{C}(G) & \text{if } s=2 \text{ and } t=-m,\\ 0 & \text{otherwise,}\end{cases}$$

and it is quite easy to make the necessary calculations;

(b) Analogously:

$$E_2^{st}=\begin{cases} \bigoplus_{i<\omega}\bigoplus_{\alpha<\omega_1} G & \text{if } s=t=0,\\ \mathbf{lim}^2\mathbf{A}^\omega(G)\approx \mathbf{lim}^1\mathbf{C}^\omega(G) & \text{if } s=2 \text{ and } t=-m,\\ 0 & \text{otherwise,}\end{cases}$$

and calculations are again easy. □

Using Theorem 1 one now obtains the following result:

Corollary 7.4. *For any $p\geqslant -2$ the homomorphism*

$$\bigoplus_{i<\omega}\overline{H}_p(X_{p+2},G)\to \overline{H}_p\left(\coprod_{i<\omega} X_{p+2},G\right)$$

is injective but not surjective.

7.1. Proof of Theorem 2

Take $X=\coprod_{p\geqslant -2} X_{p+2}$. It follows from Corollary 7.4 that

$$\bigoplus_{i<\omega}\overline{H}_p(X,G)\to \overline{H}_p\left(\coprod_{i<\omega} X,G\right)$$

is injective but not surjective for any $p\geqslant -2$.

References

[1] M. Artin, B. Mazur, Etale Homotopy, Lecture Notes in Math., vol. 100, Springer, Berlin, 1969.
[2] M.A. Batanin, On the coherent pro-homotopy category, Preprint SO AN SSSR, Novosibirsk, 1986.
[3] A. Dow, P. Simon, J.E. Vaughan, Strong homology and the proper forcing axiom, Proc. Amer. Math. Soc. 106 (3) (1989) 821–828.
[4] C.U. Jensen, Les foncteurs dérivés de $\varprojlim$ et leurs applications en théorie des modules, Lecture Notes in Math., vol. 254, Springer, Berlin, 1972.
[5] J. Lisica, S. Mardešić, Strong homology of inverse systems. III, Topology Appl. 20 (1) (1985) 29–37.
[6] J.T. Lisica, S. Mardešić, Strong homology of inverse systems of spaces. I, Topology Appl. 19 (1) (1985) 29–43.
[7] J.T. Lisica, S. Mardešić, Strong homology of inverse systems of spaces. II, Topology Appl. 19 (1) (1985) 45–64.
[8] Yu.T. Lisitsa, S. Mardešić, Steenrod homology, in: Geometric and Algebraic Topology, in: Banach Center Publ., vol. 18, PWN, Warsaw, 1986, pp. 329–343.
[9] S. Mardešić, Strong expansions and strong shape theory, Topology Appl. 38 (3) (1991) 275–291.
[10] S. Mardešić, Nonvanishing derived limits in shape theory, Topology 35 (2) (1996) 521–532.
[11] S. Mardešić, Strong Shape and Homology, Springer Monogr. Math., Springer, Berlin, 2000.
[12] J. Milnor, On axiomatic homology theory, Pacific J. Math. 12 (1962) 337–341.
[13] Z.R. Miminoshvili, The exact and semi-exact homology sequence of arbitrary spaces, Soobshch. Akad. Nauk Gruzin. SSR 113 (1) (1984) 41–44.
[14] S. Mardešić, A.V. Prasolov, Strong homology is not additive, Trans. Amer. Math. Soc. 307 (2) (1988) 725–744.
[15] B.L. Osofsky, The subscript of $\aleph_n$, projective dimension, and the vanishing of $\varprojlim^{(n)}$, Bull. Amer. Math. Soc. 80 (1974) 8–26.
[16] A.V. Prasolov, A spectral sequence for strong homology, Glas. Mat. Ser. III 24(44) (1) (1989) 17–24.
[17] A.V. Prasolov, Additivity of strong homology and the continuum hypothesis, Dokl. Akad. Nauk SSSR 310 (4) (1990) 799–802.
[18] A.V. Prasolov, Strong homology is not additive, part 2: cardinality estimates, University of Tromso, Mathematical report 98-32 (1998) 1–24.
[19] A.V. Prasolov, Strong homology is not additive, part 3: set-theoretic cohomology, University of Tromso, Mathematical Report 98-33 (1998) 1–15.
[20] A.V. Prasolov, Extraordinary strong homology, in: Geometric Topology, Dubrovnik, 1998, Topology Appl. 113 (1–3) (2001) 249–291.
[21] E.K. van Douwen, The integers and topology, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 111–167.
[22] T. Watanabe, Čech homology, Steenrod homology and strong homology. I, Glas. Mat. Ser. III 22(42) (1) (1987) 187–238.
[23] T. Watanabe, An elementary proof of the invariance of $\lim^{(n)}$ on pro-abelian groups, Glas. Mat. Ser. III 26(46) (1–2) (1991) 177–208.